Mathematical Literacy

The aim of the OECD/PISA assessment is to develop indicators of the extent to which the educational systems in participating countries have prepared 15-year-olds to play constructive roles as citizens in society. Rather than being limited to the curriculum content students have learned, the assessments focus on determining if students can use what they have learned in the situations they are likely to encounter in their daily lives.

DEFINITION OF THE DOMAIN

The OECD/PISA mathematical literacy domain is concerned with the capacities of students to analyse, reason, and communicate ideas effectively as they pose, formulate, solve, and interpret mathematical problems in a variety of situations. The OECD/PISA assessment focuses on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. In real-world settings, citizens regularly face situations when shopping, travelling, cooking, dealing with their personal finances, judging political issues, etc. in which the use of quantitative or spatial reasoning or other mathematical competencies would help clarify, formulate or solve a problem. Such uses of mathematics are based on the skills learned and practised through the kinds of problems that typically appear in school textbooks and classrooms. However, they demand the ability to apply those skills in a less structured context, where the directions are not so clear, and where the student must make decisions about what knowledge may be relevant, and how it might usefully be applied.

OECD/PISA mathematical literacy deals with the extent to which 15-year-olds can be regarded as informed, reflective citizens and intelligent consumers. Citizens in every country are increasingly confronted with a myriad of tasks involving quantitative, spatial, probabilistic or other mathematical concepts. For example, media outlets (newspapers, magazines, television, and the Internet) are filled with information in the form of tables, charts and graphs about such subjects as weather, economics, medicine and sports, to name a few. Citizens are bombarded with information on issues such as "global warming and the greenhouse effect", "population growth", "oil slicks and the seas", "the disappearing countryside". Last but not least, citizens are confronted with the need to read forms, to interpret bus and train timetables, to successfully carry out transactions involving money, to determine the best buy at the market, etc. OECD/PISA mathematical literacy focuses on the capacity of 15-year-olds (the age when many students are completing their formal compulsory mathematics learning) to use their mathematical knowledge and understanding to help make sense of these issues and to carry out the resulting tasks.

The mathematical literacy definition for OECD/PISA is:

> *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.*

Some explanatory remarks may help to further clarify this domain definition.

Mathematical literacy…

The term "mathematical literacy" has been chosen to emphasise mathematical knowledge put to functional use in a multitude of different situations in varied, reflective and insight-based ways. Of course, for such use to be possible and viable, a great deal of fundamental mathematical knowledge and skills are needed, and such skills form part of our definition of literacy. Literacy in the linguistic sense presupposes, but cannot be reduced to, a rich vocabulary and a substantial knowledge of grammatical rules, phonetics, orthography, etc. To communicate, humans combine these elements in creative ways in response to each real-world situation encountered. In the same way, mathematical literacy cannot be reduced to, but certainly presupposes, knowledge of mathematical terminology, facts and procedures, as well as skills in performing certain operations and carrying out certain methods. Mathematical literacy involves the creative combining of these elements in response to the demands imposed by the external situation.

… the world…

The term "the world" means the natural, social and cultural setting in which the individual lives. As Freudenthal (1983) stated: "Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world" (p. ix).

… to use and engage with…

The term "to use and engage with" is meant to cover using mathematics and solving mathematical problems, and also implies a broader personal involvement through *communicating, relating to, assessing* and even *appreciating and enjoying* mathematics. Thus the definition of mathematical literacy encompasses the functional use of mathematics in a narrow sense as well as preparedness for further study, and the aesthetic and recreational elements of mathematics.

… that individual's life…

The phrase "that individual's life" includes his or her private life, occupational life, and social life with peers and relatives, as well as life as a citizen of a community.

A crucial capacity implied by this notion of mathematical literacy is the ability to pose, formulate, solve, and interpret problems using mathematics within a variety of situations or contexts. The contexts range from purely mathematical ones to contexts in which no mathematical structure is present or apparent at the outset – the problem poser or solver must successfully introduce the mathematical structure. It is also important to emphasise that the definition is not just concerned with knowing mathematics at some minimal level; it is also about doing and using mathematics in situations that range from the everyday to the unusual, from the simple to the complex.

Mathematics related attitudes and emotions such as self-confidence, curiosity, feelings of interest and relevance, and the desire to do or understand things, are not components of the definition of mathematical literacy but nevertheless are important contributors to it. In principle it is possible to possess mathematical literacy without possessing such attitudes and emotions. In practice, however, it is not likely that such literacy is going to be exerted and put into practice by someone who does not have some degree of self-confidence, curiosity, feelings of interest and relevance, and the desire to do or understand things that contain mathematical components. The importance of these attitudes and emotions as correlates of mathematical literacy is recognised. They are not part of the mathematical literacy assessment, but will be addressed in other components of OECD/PISA.

THEORETICAL BASIS FOR THE OECD/PISA MATHEMATICS FRAMEWORK

The OECD/PISA definition of mathematical literacy is consistent with the broad and integrative theory about the structure and use of language as reflected in recent socio-cultural literacy studies. In James Gee's *Preamble to a Literacy Program* (1998), the term "literacy" refers to the human use of language. The ability to read, write, listen and speak a language is the most important tool through which human social activity is mediated. In fact, each human language and use of language has an intricate design tied in complex ways to a variety of functions. For a person to be literate in a language implies that the person knows many of the design resources of the language and is able to use those resources for several different social *functions*. Analogously, considering mathematics as a language implies that students must learn the design features involved in mathematical discourse (the terms, facts, signs and symbols, procedures and skills in performing certain operations in specific mathematical sub-domains, and the structure of those ideas in each sub-domain), and they also must learn to use such ideas to solve non-routine problems in a variety of situations defined in terms of social functions. Note that the design features for mathematics include knowing the basic terms, procedures and concepts commonly taught in schools, and also involve knowing how these features are structured and used. Unfortunately, one can know a good deal about the design features of mathematics without knowing either their structure or how to use those features to solve problems. These scholarly notions involving the interplay of "design features" and "functions" that support the mathematics framework for OECD/PISA can be illustrated via the following example.

Mathematics Example 1: Streetlight

The Town Council has decided to construct a streetlight in a small triangular park so that it illuminates the whole park. Where should it be placed?

This social problem can be solved by following the general strategy used by mathematicians, which the mathematics framework will refer to as *mathematising*. Mathematising can be characterised as having five aspects:

1. Starting with a problem situated in reality;

   Locating where a street light is to be placed in a park.

2. Organising it according to mathematical concepts;

   The park can be represented as a triangle, and illumination from a light as a circle with the street light at its centre.

3. Gradually trimming away the reality through processes such as making assumptions about which features of the problem are important, generalising and formalising (which promote the mathematical features of the situation and transform the real problem into a mathematical problem that faithfully represents the situation);

   The problem is transformed into locating the centre of a circle that circumscribes the triangle.

4. Solving the mathematical problem; and

   Using the fact that the centre of a circle that circumscribes a triangle lies at the point of intersection of the perpendicular bisectors of the triangle's sides, construct the perpendicular bisectors of two sides of the triangle. The point of intersection of the bisectors is the centre of the circle.

5. Making sense of the mathematical solution in terms of the real situation.

   Relating this finding to the real park. Reflecting on this solution and recognising, for example, that if one of the three corners of the park were an obtuse angle, this solution would not be reasonable since the location of the light would be outside the park. Recognising that the location and size of trees in the park are other factors affecting the usefulness of the mathematical solution.

It is these processes that characterise how, in a broad sense, mathematicians often *do mathematics*, how people use mathematics in a variety of current and potential occupations, and how informed and reflective citizens should use mathematics to fully and competently engage with the real world. In fact, learning to *mathematise* should be a primary educational goal for all students.

Today and in the foreseeable future, every country needs mathematically literate citizens to deal with a very complex and rapidly changing society. Accessible information has been growing exponentially, and citizens need to be able to decide how to deal with this information. Social debates increasingly involve quantitative information to support claims. One example of the need for mathematical literacy is the frequent demand for individuals to make judgements and assess the accuracy of conclusions and claims in surveys and studies. Being able to judge the soundness of the claims from such arguments is, and increasingly will be, a critical aspect of being a responsible citizen. The steps of the mathematisation process discussed in this framework are the fundamental elements of using mathematics in such complex situations. Failure to use mathematical notions can result in confused personal decisions, an increased susceptibility to pseudo-sciences, and poorly informed decision-making in professional and public life.

A mathematically literate citizen realises how quickly change is taking place and the consequent need to be open to lifelong learning. Adapting to these changes in a creative, flexible and practical way is a necessary condition for successful citizenship. The skills learned at school will probably not be sufficient to serve the needs of citizens for the majority of their adult life.

The requirements for competent and reflective citizenship also affect the workforce. Workers are less and less expected to carry out repetitive physical chores for all of their working lives. Instead, they are engaged actively in monitoring output from a variety of high-technology machines, dealing with a flood of information, and engaging in team problem solving. The trend is that more and more occupations will require the ability to understand, communicate, use and explain concepts and procedures based on mathematical thinking. The steps of the mathematisation process are the building blocks of this kind of mathematical thinking.

Finally, mathematically literate citizens also develop an appreciation for mathematics as a dynamic, changing and relevant discipline that may often serve their needs.

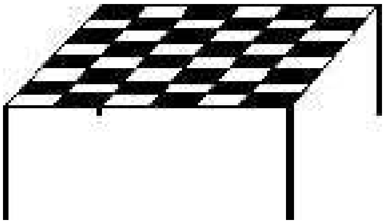
The operational problem faced by OECD/PISA is how to assess whether 15-year-old students are mathematically literate in terms of their ability to *mathematise*. Unfortunately, in a timed assessment this is difficult because for most complex real situations the full process of proceeding from reality to mathematics and back often involves collaboration and finding appropriate resources, and takes considerable time.

To illustrate *mathematisation* in an extended problem-solving exercise, consider the "Fairground" Example 2 carried out by an eighth grade class of students (Romberg, 1994):

Mathematics Example 2: FAIRGROUND GAMEBOARD

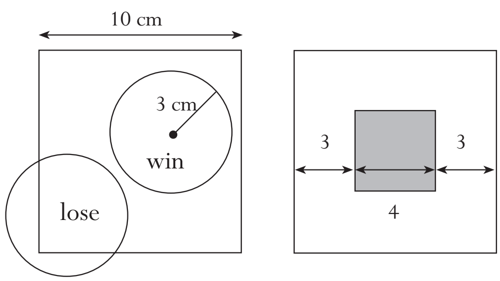
At a fair, players throw coins onto a board chequered with squares. If a coin touches a boundary, it is lost. If it rolls off the board, it is returned. But if the coin lies wholly within a square, the player wins the coin back plus a prize.

What is the probability of winning at this game?



Clearly this exercise is situated in reality. First, the students began by realising that the probability of winning depends on the relative sizes of the squares and the coin (identifying the important variables). Next, to transform the real problem into a mathematical problem, they realised that it might be better to examine the relationship for a single square and a smaller circle (trimming the reality). Then they decided to construct a specific example (using a problem solving heuristic – "if you cannot solve the problem given, solve one you can"). Note that all the following work was done with respect to this specific example, not the board, the prize, etc. In the example, they let the radius of the coin be 3 cm and the side of the squares be 10 cm. They realised that to win, the centre of the coin must be at least 3 cm from each side; otherwise the edge of the coin will fall across the square. The sample space was the square with side 10 cm, and the winning event space was a square with side 4 cm. The relationships are shown in the following diagram (Figure 1.1).

Figure 1.1 ■ **A winning toss and a losing toss (on the left) and the sample and event spaces (on the right)**



The probability of winning was obtained from the ratio of the area of the sample and event space squares (for the example p = 16/100). Then the students examined coins of other sizes, and generalised the problem by expressing its solution in algebraic terms. Finally the students extended this finding to work out the relative sizes of the coin and squares for a variety of practical situations; they constructed boards and empirically tested results (making sense of the mathematical solution in terms of the real situation).

Note that each of the five aspects of *mathematisation* is apparent in this solution. Although the problem is complex, all 15-year-old students should understand the mathematical features needed to solve the problem. However, note that in this class the students worked together on this exercise for three days.

Ideally, to judge whether 15-year-old students can use their accumulated mathematical knowledge to solve mathematical problems they encounter in their world, one would collect information about their ability to *mathematise* such complex situations. Clearly this is impractical. Instead, OECD/PISA has chosen to prepare items to assess different parts of this process. The following section describes the strategy chosen to create a set of test items in a balanced manner so that a selected sample of these items will cover the five aspects of *mathematising*. The aim is to use the responses to those items to locate students on a scale of proficiency in the OECD/PISA construct of mathematical literacy.

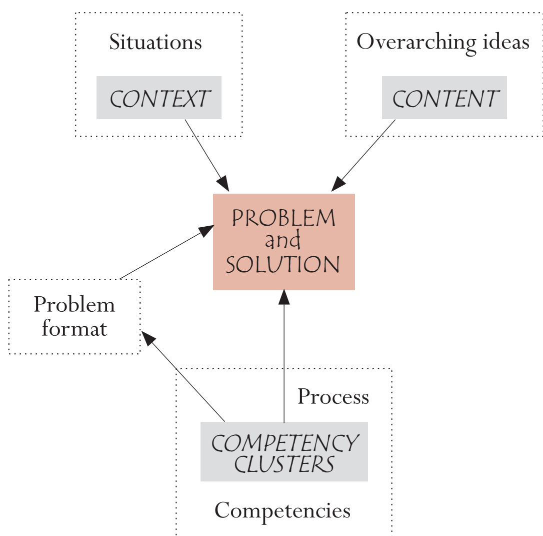
ORGANISATION OF THE DOMAIN

The OECD/PISA mathematics framework provides the rationale for, and the description of, an assessment of the extent to which 15-year-olds can handle mathematics in a well-founded manner when confronted with real-world problems, or, in more general terms, an assessment of how mathematically literate 15-year-olds are. To describe the domain that is assessed more clearly, three components must be distinguished:

- the *situations or contexts* in which the problems are located,
- the *mathematical content* that has to be used to solve the problems, organised by certain overarching *ideas*, and, most importantly,
- the *competencies* that have to be activated in order to connect the real world, in which the problems are generated, with mathematics, and thus to solve the problems.

These components are represented in visual form in Figure 1.2. An explanation of each is provided afterwards.

Figure 1.2 ■ **The components of the mathematics domain**



The extent of a person's mathematical literacy is seen in the way he or she uses mathematical knowledge and skills in solving problems. Problems (and their solution) may occur in a variety of "situations or contexts" within the experience of an individual. OECD/PISA problems draw from the real world in two ways. First, problems exist within some broad situations that are relevant to the student's life. The situations form part of the real world and are indicated by a big square in the upper left of the picture. Next, within that situation, problems have a more specific context. This is represented by the small square within the situations square.

In the above examples the situation is the local community, and the contexts are lighting in a park (Example 1), and a fairground checkerboard game (Example 2).

The next component of the real world that has to be considered when thinking about mathematical literacy is the mathematical content that a person might bring to bear in solving a problem. The *mathematical content* can be illustrated by four categories that encompass the kinds of problems that arise through interaction with day-to-day phenomena, and that are based on a conception of the ways in which mathematical content presents itself to people. For PISA assessment purposes, these are called "overarching ideas": *quantity*, *space and shape*, *change and relationships*, and *uncertainty*. This is somewhat different from an approach to content that would be familiar from the perspective of mathematics instruction and the curricular strands typically taught in schools. However, the overarching ideas together broadly encompass the range of mathematical topics that students are expected to have learned. The overarching ideas are represented by the big square in the upper right of the diagram in Figure 1.2. From the overarching ideas the content used in solving a problem is extracted. This is represented by the smaller square within the overarching ideas square.

The arrows going from the "context" and "content" to the problem show how the real world (including mathematics) makes up a problem.

The park problem (Example 1) involves geometrical knowledge related to the ideas of space and shape, and the fairground problem (Example 2) involves (at least in its initial stages) dealing with uncertainty and applying knowledge of probability.

The mathematical processes that students apply as they attempt to solve problems are referred to as *mathematical competencies*. Three *competency clusters* encapsulate the different cognitive processes that are needed to solve various kinds of problems. These clusters reflect the way that mathematical processes are typically employed when solving problems that arise as students interact with their world, and will be described in detail in later sections.

Thus the process component of this framework is represented in first by the large square, representing the general mathematical competencies, and a smaller square that represents the three competency clusters. The particular competencies needed to solve a problem will be related to the nature of the problem, and the competencies used will be reflected in the solution found. This interaction is represented by the arrow from the competency clusters to the problem and its solution.

The remaining arrow goes from the competency clusters to the problem format. The competencies employed in solving a problem are related to the form of the problem and its precise demands.

It should be emphasised that the three components just described are of different natures. While situations or contexts define the real-world problem areas, and overarching ideas reflect the way in which we look at the world with "mathematical glasses", the competencies are the core of mathematical literacy. Only when certain competencies are available to students will they be in a position to successfully solve given problems. Assessing mathematical literacy includes assessing to what extent students possess mathematical competencies they can productively apply in problem situations.

In the following sections, these three components are described in more detail.

Situations or contexts

An important aspect of mathematical literacy is engagement with mathematics: using and doing mathematics in a variety of situations. It has been recognised that in dealing with issues that lend themselves to a mathematical treatment, the choice of mathematical methods and representations is often dependent on the situations in which the problems are presented.

The situation is the part of the student's world in which the tasks are placed. It is located at a certain distance from the students. For OECD/PISA, the closest situation is the student's personal life; next is school life, work life and leisure, followed by the local community and society as encountered in daily life. Furthest away are scientific situations. Four situation-types will be defined and used for problems to be solved: personal, educational/occupational, public, and scientific.

The context of an item is its specific setting within a situation. It includes all the detailed elements used to formulate the problem.

Consider the following example:

Mathematics Example 3: SAVINGS ACCOUNT

1000 zed is put into a savings account at a bank. There are two choices: one can get an annual rate of 4% OR one can get an immediate 10 zed bonus from the bank, and a 3% annual rate. Which option is better after one year? After two years?

The situation of this item is "finance and banking", which is a situation from the local community and society that OECD/PISA would classify as "public". The context of this item concerns money (zeds) and interest rates for a bank account.

Note that this kind of problem is one that could be part of the actual experience or practice of the participant in some real-world setting. It provides an *authentic* context for the use of mathematics, since the application of mathematics in this context would be genuinely directed to solving the problem[1]. This can be contrasted with problems frequently seen in school mathematics texts, where the main purpose is to practise the mathematics involved rather than to use mathematics to solve a real problem. This *authenticity* in the use of mathematics is an important aspect of the design and analysis of items for OECD/PISA, strongly related to the definition of mathematical literacy.

It should also be noted that there are some made-up elements of the problem – the money involved is fictitious. This fictitious element is introduced to ensure that students from certain countries are not given an unfair advantage.

The situation and context of a problem can also be considered in terms of the distance between the problem and the mathematics involved. If a task refers only to mathematical objects, symbols or structures, and makes no reference to matters outside the mathematical world, the context of the task is considered as intra-mathematical, and the task will be classified as belonging to the "scientific" situation-type. A limited range of such tasks will be included in OECD/PISA, where the close link between the problem and the underlying mathematics is made explicit in the problem context. More typically, problems encountered in the day-to-day experience of the student are not stated in explicit mathematical terms. They refer to real-world objects. These task contexts are called "extra-mathematical", and the student must translate these problem contexts into a mathematical form. Generally speaking, OECD/PISA puts an emphasis on tasks that might be encountered in some real-world situation and possess an authentic context for the use of mathematics that influences the solution and its interpretation. Note that this does not preclude the inclusion of tasks in which the context is hypothetical, as long as the context has some real elements, is not too far removed from a real-world situation, and for which the use of mathematics to solve the problem would be authentic. Example 4 shows a problem with a hypothetical context that is "extra-mathematical":

Mathematics Example 4: COINAGE SYSTEM

Would it be possible to establish a coinage system based on only the denominations 3 and 5? More specifically, what amounts could be reached on that basis? Would such a system be desirable?

This problem derives its quality not primarily from its closeness to the real world, but from the fact that it is mathematically interesting and calls on competencies that are related to mathematical literacy. The use of mathematics to explain hypothetical scenarios and explore potential systems or situations, even if these are unlikely to be carried out in reality, is one of its most powerful features. Such a problem would be classified as belonging to the "Scientific" situation-type.

1. Note that this use of the term "authentic" is not intended to indicate that mathematics items are in some sense genuine and real. OECD/PISA mathematics uses the terms "authentic" to indicate that the use of mathematics is genuinely directed to solving the problem at hand, rather than the problem being merely a vehicle for the purpose of practising some mathematics.

In summary, OECD/PISA places most value on tasks that could be encountered in a variety of real-world situations, and that have a context in which the use of mathematics to solve the problem would be authentic. Problems with extra-mathematical contexts that influence the solution and its interpretation are preferred as a vehicle for assessing mathematical literacy, since these problems are most like those encountered in day-to-day life.

Mathematical content – The four "overarching ideas"

Mathematical concepts, structures and ideas have been invented as tools to organise the phenomena of the natural, social and mental world. In schools, the mathematics curriculum has been logically organised around content strands (*e.g.*, arithmetic, algebra, geometry) and their detailed topics that reflect historically well-established branches of mathematical thinking, and that facilitate the development of a structured teaching syllabus. However, in the real world the phenomena that lend themselves to mathematical treatment do not come so logically organised. Rarely do problems arise in ways and contexts that allow their understanding and solution to be achieved through an application of knowledge from a single content strand. The "fairground" problem described in Example 2 provides an example of a problem that draws on quite diverse mathematical areas.

Since the goal of OECD/PISA is to assess students' capacity to solve real problems, our strategy has been to define the range of content that will be assessed using a phenomenological approach to describing the mathematical concepts, structures or ideas. This means describing content in relation to the phenomena and the kinds of problems for which it was created. This approach ensures a focus in the assessment that is consistent with the domain definition, yet covers a range of content that includes what is typically found in other mathematics assessments and in national mathematics curricula.

A phenomenological organisation for mathematical content is not new. Two well known publications: *On the shoulders of giants: New approaches to numeracy* (Steen, 1990) and *Mathematics: The science of patterns* (Devlin, 1994) have described mathematics in this manner. However, various ways of labelling the approach and naming the different phenomenological categories have been used. Suggestions for labelling the approach have included "deep ideas", "big ideas", or "fundamental ideas"; "overarching concepts", "overarching ideas", "underlying concepts" or "major domains"; or "problematique". In the mathematics framework for OECD/PISA 2003, the label "overarching ideas" will be used.

There are many possible mathematical overarching ideas. The above mentioned publications alone refer to pattern, dimension, quantity, uncertainty, shape, change, counting, reasoning and communication, motion and change, symmetry and regularity, and position. Which should be used for the OECD/PISA mathematics framework? For the purpose of focusing the mathematical literacy domain, it is important to make a selection of problem areas that grows out of historical

developments in mathematics, that encompasses sufficient variety and depth to reveal the essentials of mathematics, and that also represents or includes the conventional mathematical curricular strands in an acceptable way.

For centuries mathematics was predominantly the science of numbers, together with relatively concrete geometry. The period up to 500 BC in Mesopotamia, Egypt and China saw the origin of the concept of number. Operations with numbers and quantities, including quantities resulting from geometrical measurements, were developed. From 500 BC to 300 AD was the era of Greek mathematics, which focused primarily on the study of geometry as an axiomatic theory. The Greeks were responsible for redefining mathematics as a unified science of number and shape. The next major change took place between 500 and 1300 AD in the Islamic world, India and China, which established algebra as a branch of mathematics. This founded the study of relationships. With the independent inventions of calculus (the study of change, growth and limit) by Newton and Leibniz in the 17th century, mathematics became an integrated study of number, shape, change and relationships.

The 19th and 20th centuries saw explosions of mathematical knowledge and of the range of phenomena and problems that could be approached by means of mathematics. These include aspects of randomness and indeterminacy. These developments made it increasingly difficult to give simple answers to the question "what is mathematics?" At the time of the new millennium, many see mathematics as the science of patterns (in a general sense). Thus, a choice of overarching ideas can be made that reflects these developments: patterns in *quantity*, patterns in *space and shape*, patterns in *change and relationships* form central and essential concepts for any description of mathematics, and they form the heart of any curriculum, whether at high school, college or university. But to be literate in mathematics means more. Dealing with uncertainty from a mathematical and scientific perspective is essential. For this reason, elements of probability theory and statistics give rise to the fourth overarching idea: *uncertainty*.

The following list of overarching ideas, therefore, is used in OECD/PISA 2003 to meet the requirements of historical development, coverage of the domain, and reflection of the major threads of school curriculum:

- *quantity;*
- *space and shape;*
- *change and relationships;*
- *uncertainty.*

With these four, mathematical content is organised into a sufficient number of areas to ensure a spread of items across the curriculum, but at the same time a number small enough to avoid a too fine division that would work against a focus on problems based in real situations.

The basic conception of an overarching idea is an encompassing set of phenomena and concepts that make sense and can be encountered within and across a multitude of different situations. By its very nature, each overarching idea can be perceived as a sort of general notion dealing with some generalised content dimension. This implies that the overarching ideas cannot be sharply delineated vis-à-vis one another[2]. Rather, each of them represents a certain perspective, or point of view, which can be thought of as possessing a core, a centre of gravity, and somewhat blurred outskirts that allow for intersection with other overarching ideas. In principle, any overarching idea intersects any other overarching idea. The four overarching ideas are summarised in the following section and discussed more fully afterwards.

Quantity

This overarching idea focuses on the need for quantification in order to organise the world. Important aspects include an understanding of relative size, the recognition of numerical patterns, and the use of numbers to represent quantities and quantifiable attributes of real-world objects (counts and measures). Furthermore, *quantity* deals with the processing and understanding of numbers that are represented to us in various ways.

An important aspect of dealing with *quantity* is quantitative reasoning. Essential components of quantitative reasoning are number sense, representing numbers in various ways, understanding the meaning of operations, having a feel for the magnitude of numbers, mathematically elegant computations, mental arithmetic and estimating.

Space and shape

Patterns are encountered everywhere: in spoken words, music, video, traffic, building constructions and art. Shapes can be regarded as patterns: houses, office buildings, bridges, starfish, snowflakes, town plans, cloverleaves, crystals and shadows. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels (Grünbaum, 1985).

The study of shape and constructions requires looking for similarities and differences when analysing the components of form and recognising shapes in different representations and different dimensions. The study of shapes is closely connected to the concept of "grasping space". This means learning to know, explore and conquer, in order to live, breathe and move with more understanding in the space in which we live (Freudenthal, 1973).

To achieve this requires understanding the properties of objects and their relative positions. We must be aware of how we see things and why we see them as we do. We must learn to navigate through space and through constructions and shapes. This means understanding the relationship between shapes and images or visual representations, such as that between a real city and photographs and maps of the same city. It includes also understanding how

2. And, of course, neither can traditional mathematics content strands.

three-dimensional objects can be represented in two dimensions, how shadows are formed and must be interpreted, what perspective is and how it functions.

Change and relationships

Every natural phenomenon is a manifestation of change, and the world around us displays a multitude of temporary and permanent relationships among phenomena. Examples are organisms changing as they grow, the cycle of seasons, the ebb and flow of tides, cycles of unemployment, weather changes and stock exchange indices. Some of these change processes involve and can be described or modelled by straightforward mathematical functions: linear, exponential, periodic or logistic, either discrete or continuous. But many relationships fall into different categories, and data analysis is often essential to determine the kind of relationship that is present. Mathematical relationships often take the shape of equations or inequalities, but relations of a more general nature (*e.g.*, equivalence, divisibility, inclusion, to mention but a few) may appear as well.

Functional thinking – that is, thinking in terms of and about relationships – is one of the most fundamental disciplinary aims of the teaching of mathematics (MAA, 1923). Relationships may be given a variety of different representations, including symbolic, algebraic, graphical, tabular and geometrical. Different representations may serve different purposes and have different properties. Hence translation between representations often is of key importance in dealing with situations and tasks.

Uncertainty

The present "information society" offers an abundance of information, often presented as accurate, scientific and with a degree of certainty. However, in daily life we are confronted with uncertain election results, collapsing bridges, stock market crashes, unreliable weather forecasts, poor predictions for population growth, economic models that don't align, and many other demonstrations of the uncertainty of our world.

Uncertainty is intended to suggest two related topics: data and chance. These phenomena are respectively the subject of mathematical study in statistics and probability. Relatively recent recommendations concerning school curricula are unanimous in suggesting that statistics and probability should occupy a much more prominent place than has been the case in the past (Committee of Inquiry into the Teaching of Mathematics in Schools, 1982; LOGSE, 1990; MSEB, 1990; NCTM, 1989; NCTM, 2000).

Specific mathematical concepts and activities that are important in this area are collecting data, data analysis and display/visualisation, probability and inference.

We now turn to the most important aspect of the mathematics framework: a discussion of the competencies that students bring to bear when attempting to solve problems. These are discussed under the broad heading of mathematical processes.

Mathematical processes

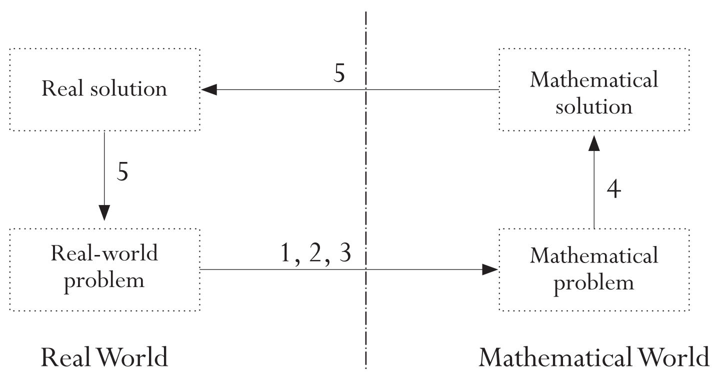
Introduction – Mathematisation

OECD/PISA examines the capacities of students to analyse, reason, and communicate mathematical ideas effectively as they pose, formulate, solve and interpret mathematical problems in a variety of situations. Such problem solving requires students to use the skills and competencies they have acquired through schooling and life experiences. In OECD/PISA, a fundamental process that students use to solve real-life problems is referred to as "mathematisation".

> *Newton might have been describing mathematisation in his major work, "Mathematical Principles of Natural Philosophy" when he wrote: "But our purpose is only to trace out the quantity and properties of this force from the phenomena, and to apply what we discover in some simple cases as principles, by which, in a mathematical way, we may estimate the effects thereof in more involved cases" (Newton, 1687).*

The earlier discussion of the theoretical basis for the OECD/PISA mathematics framework outlined a five-step description of mathematisation. These steps are shown in Figure 1.3.

Figure 1.3 ■ **The mathematisation cycle**



(1) Starting with a problem situated in reality;

(2) Organising it according to mathematical concepts and identifying the relevant mathematics;

(3) Gradually trimming away the reality through processes such as making assumptions, generalising and formalising, which promote the mathematical features of the situation and transform the real-world problem into a mathematical problem that faithfully represents the situation;

(4) Solving the mathematical problem; and

(5) Making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution.

As the diagram in Figure 1.3 suggests, the five aspects will be discussed in three stages.

Mathematisation first involves translating the problem from "reality" into mathematics. This process includes activities such as:

- identifying the relevant mathematics with respect to a problem situated in reality;
- representing the problem in a different way; including organising it according to mathematical concepts and making appropriate assumptions;
- understanding the relationships between the language of the problem, and symbolic and formal language needed to understand it mathematically;
- finding regularities, relations, and patterns;
- recognising aspects that are isomorphic with known problems;
- translating the problem into mathematics; *i.e.*, to a mathematical model, (de Lange, 1987, p. 43).

As soon as a student has translated the problem into a mathematical form, the whole process can continue within mathematics. Students will pose questions like: "Is there…?", "If so, how many?", "How do I find…?", using known mathematical skills and concepts. They will attempt to work on their model of the problem situation, to adjust it, to establish regularities, to identify connections and to create a good mathematical argument. This part of the mathematisation process is generally called the deductive part of the modelling cycle (Blum, 1996; Schupp, 1988). However, other than strictly deductive processes may play a part in this stage. This part of the mathematisation process includes:

- using and switching between different representations;
- using symbolic, formal and technical language and operations;
- refining and adjusting mathematical models; combining and integrating models;
- argumentation;
- generalisation.

The last step or steps in solving a problem involve reflecting on the whole mathematisation process and the results. Here students must interpret the results with a critical attitude and validate the whole process. Such reflection takes place at all stages of the process, but it is especially important at the concluding stage. Aspects of this reflecting and validating process are:

- understanding the extent and limits of mathematical concepts;
- reflecting on mathematical arguments, and explaining and justifying results;
- communicating the process and solution;
- critiquing the model and its limits.

This stage is indicated in two places in Figure 1.3 by the label "5", where the mathematisation process passes from the mathematical solution to the real solution, and where this is related back to the original real-world problem.

The competencies

The previous section focused on the major concepts and processes involved in mathematisation. An individual who is to engage successfully in mathematisation within a variety of situations, extra- and intra-mathematical contexts, and overarching ideas, needs to possess a number of mathematical competencies which, taken together, can be seen as constituting comprehensive mathematical competence. Each of these competencies can be possessed at different levels of mastery. Different parts of mathematisation draw differently upon these competencies, both in regard to the particular ones involved and in regard to the required level of mastery. To identify and examine these competencies, OECD/PISA has decided to make use of eight characteristic mathematical competencies that rely, in their present form, on the work of Niss (1999) and his Danish colleagues. Similar formulations may be found in the work of many others (as indicated in Neubrand *et al.*, 2001). Some of the terms used, however, have different usage among different authors.

1. *Thinking and reasoning*. This involves posing questions characteristic of mathematics ("Is there…?", "If so, how many?", "How do we find…?"); knowing the kinds of answers that mathematics offers to such questions; distinguishing between different kinds of statements (definitions, theorems, conjectures, hypotheses, examples, conditioned assertions); and understanding and handling the extent and limits of given mathematical concepts.

2. *Argumentation*. This involves knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning; following and assessing chains of mathematical arguments of different types; possessing a feel for heuristics ("What can(not) happen, and why?"); and creating and expressing mathematical arguments.

3. *Communication*. This involves expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others' written or oral statements about such matters.

4. *Modelling*. This involves structuring the field or situation to be modelled; translating "reality" into mathematical structures; interpreting mathematical models in terms of "reality"; working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating about the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.

5. *Problem posing and solving*. This involves posing, formulating, and defining different kinds of mathematical problems (for example "pure", "applied", "open-ended" and "closed"); and solving different kinds of mathematical problems in a variety of ways.

6. *Representation*. This involves decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations, and the interrelationships between the various representations; choosing and switching between different forms of representation, according to situation and purpose.

7. *Using symbolic, formal and technical language and operations*. This involves decoding and interpreting symbolic and formal language, and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; using variables, solving equations and undertaking calculations.

8. *Use of aids and tools*. This involves knowing about, and being able to make use of, various aids and tools (including information technology tools) that may assist mathematical activity, and knowing about the limitations of such aids and tools.

OECD/PISA does not intend to develop test items that assess the above competencies individually. There is considerable overlap among them, and when using mathematics, it is usually necessary to draw simultaneously on many of the competencies, so that any effort to assess individual ones is likely to result in artificial tasks and unnecessary compartmentalisation of the mathematical literacy domain. The particular competencies students will be able to display will vary considerably among individuals. This is partially because all learning occurs through experiences, "with individual knowledge construction occurring through the processes of interaction, negotiation, and collaboration" (de Corte, Greer, & Verschaffel, 1996, p. 510). OECD/PISA assumes that much of students' mathematics is learned in schools. Understanding of a domain is acquired gradually. More formal and abstract ways of representing and reasoning emerge over time as a consequence of engagement in activities designed to help informal ideas evolve. Mathematical literacy is also acquired through experience involving interactions in a variety of social situations or contexts.

In order to productively describe and report students' capabilities, as well as their strengths and weaknesses from an international perspective, some structure is needed. One way of providing this in a comprehensible and manageable way is to describe clusters of competencies, based on the kinds of cognitive demands needed to solve different mathematical problems.

Competency clusters

OECD/PISA has chosen to describe the cognitive activities that these competencies encompass according to three *competency clusters*: the *reproduction* cluster, the *connections* cluster, and the *reflection* cluster. In the following sections the three clusters are described, and the ways in which the individual competencies are played out in each cluster are discussed.

The reproduction cluster

The competencies in this cluster essentially involve reproduction of practised knowledge. They include those most commonly used on standardised assessments and classroom tests. These competencies are knowledge of facts and of common problem representations, recognition of equivalents, recollection of familiar mathematical objects and properties, performance of routine procedures, application of standard algorithms and technical skills, manipulation of expressions containing symbols and formulae in standard form, and carrying out computations.

1. *Thinking and reasoning*. This involves posing the most basic forms of questions ("how many…?", "how much is…?") and understanding the corresponding kinds of answers ("so many…", "this much…"); distinguishing between definitions and assertions; understanding and handling mathematical concepts in the sorts of contexts in which they were first introduced or have subsequently been practised.

2. *Argumentation*. This involves following and justifying standard quantitative processes, including computational processes, statements and results.

3. *Communication*. This involves understanding and expressing oneself orally and in writing about simple mathematical matters, such as reproducing the names and the basic properties of familiar objects, citing computations and their results, usually not in more than one way.

4. *Modelling*. This involves recognising, recollecting, activating, and exploiting well structured familiar models; interpreting back and forth between such models (and their results) and "reality"; and elementary communication about model results.

5. *Problem posing and solving*. This involves posing and formulating problems by recognising and reproducing practised standard pure and applied problems in closed form; and solving such problems by invoking and using standard approaches and procedures, typically in one way only.

6. *Representation*. This involves decoding, encoding and interpreting familiar, practised standard representations of well known mathematical objects. Switching between representations is involved only when the switching itself is an established part of the representations implied.

7. *Using symbolic, formal and technical language and operations*. This involves decoding and interpreting routine basic symbolic and formal language practised in well known contexts and situations; handling simple statements and expressions containing symbols and formulae, including using variables, solving equations and undertaking calculations by routine procedures.

8. *Use of aids and tools*. This involves knowing about and being able to use familiar aids and tools in contexts, situations and ways close to those in which their use was introduced and practised.

Assessment items measuring the *reproduction* cluster competencies might be described with the following key descriptors: reproducing practised material and performing routine operations.

Examples of reproduction cluster items

Mathematics Example 5

Solve the equation 7x–3 = 13x + 15

Mathematics Example 6

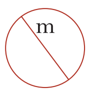
What is the average of 7, 12, 8, 14, 15, 9?

Mathematics Example 7

Write 69% as a fraction.

Mathematics Example 8

Line m is called the circle's: ________



Mathematics Example 9

1 000 zed is put in a savings account at a bank, with an interest rate of 4%. How many zed will there be in the account after one year?

In order to clarify the boundary for items from the *reproduction* cluster, the savings account problem described in Example 3 provided an example that does NOT belong to the *reproduction* cluster. This problem will take most students beyond the simple application of a routine procedure, and requires the application of a chain of reasoning and a sequence of computational steps that are not characteristic of the *reproduction* cluster competencies.

The connections cluster

The *connections* cluster competencies build on the *reproduction* cluster competencies in taking problem solving to situations that are not simply routine, but still involve familiar, or quasi-familiar, settings.

In addition to the competencies described for the *reproduction* cluster, for the *connections* cluster the competencies include the following:

1. *Thinking and reasoning*. This involves posing questions ("how do we find…?", "which mathematics is involved…?") and understanding the corresponding kinds of answers (provided by means of tables, graphs, algebra, figures, etc.); distinguishing between definitions and assertions, and between different kinds of assertions; and understanding and handling mathematical concepts in contexts that are slightly different from those in which they were first introduced or have subsequently been practised.

2. *Argumentation*. This involves simple mathematical reasoning without distinguishing between proofs and broader forms of argument and reasoning;

following and assessing chains of mathematical arguments of different types, and possessing a feel for heuristics (*e.g.* "what can or cannot happen, or be the case, and why?", "what do we know, and what do we want to obtain?").

3. *Communication*. This involves understanding and expressing oneself orally and in writing about mathematical matters ranging from reproducing the names and basic properties of familiar objects and explaining computations and their results (usually in more than one way), to explaining matters that include relationships. It also involves understanding others' written or oral statements about such matters.

4. *Modelling*. This involves structuring the field or situation to be modelled; translating "reality" into mathematical structures in contexts that are not too complex but nevertheless different from what students are usually familiar with. It involves also interpreting back and forth between models (and their results) and "reality", including aspects of communication about model results.

5. *Problem posing and solving*. This involves posing and formulating problems beyond the reproduction of practised standard pure and applied problems in closed form; solving such problems by invoking and using standard approaches and procedures, but also more independent problem solving processes in which connections are made between different mathematical areas and modes of representation and communication (schemata, tables, graphs, words, pictures).

6. *Representation*. This involves decoding, encoding and interpreting familiar and less familiar representations of mathematical objects; choosing and switching between different forms of representation of mathematical objects and situations, and translating and distinguishing between different forms of representation.

7. *Using symbolic, formal and technical language and operations*. This involves decoding and interpreting basic symbolic and formal language in less well known contexts and situations, and handling statements and expressions containing symbols and formulae, including using variables, solving equations and undertaking calculations by familiar procedures.

8. *Use of aids and tools*. This involves knowing about and using familiar aids and tools in contexts, situations and ways that are different from those in which their use was introduced and practised.

Items associated with this cluster usually require some evidence of the integration and connection of material from the various overarching ideas, or from different mathematical curriculum strands, or the linking of different representations of a problem.

Assessment items measuring the *connections* cluster of competencies might be described with the following key descriptors: integrating, connecting, and modest extension of practised material.

Examples of connections cluster items

A first example of a *connections* cluster item was given in the "savings account" problem described in Example 3. Other examples of *connections* cluster items follow.

Mathematics Example 10: DISTANCE

Mary lives two kilometres from school, Martin five.

How far do Mary and Martin live from each other?

When this problem was originally presented to teachers, many of them rejected it on the ground that it was too easy – one could easily see that the answer is 3. Another group of teachers argued that this was not a good item because there was no answer – meaning there is not one single numerical answer. A third reaction was that it was not a good item because there were many possible answers, since without further information the most that can be concluded is that they live somewhere between 3 and 7 kilometres apart, and that is not desirable for an item. A small group thought it was an excellent item, because you have to understand the question, it is real problem solving because there is no strategy known to the student, and it is beautiful mathematics, although you have no clue how students will solve the problem. It is this last interpretation that associates the problem with the *connections* cluster of competencies.

Mathematics Example 11: THE OFFICE RENTING

The following two advertisements appeared in a daily newspaper in a country where the units of currency are zeds.

BUILDING A	*BUILDING B*
Office space available	Office space available
58-95 square metres 475 zeds per month	35-260 square metres 90 zeds per square metre per year
100-120 square metres 800 zeds per month	

If a company is interested in renting an office of 110 square metres in that country for a year, at which office building, A or B, should the company rent the office in order to get the lower price? Show your work. [© IEA/TIMSS]

Mathematics Example 12: THE PIZZA

A pizzeria serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds. [© PRIM, Stockholm Institute of Education]

Which pizza is better value for money? Show your reasoning.

In both of these problems, students are required to translate a real-world situation into mathematical language, to develop a mathematical model that enables them to make a suitable comparison, to check that the solution fits in with the initial question context and to communicate the result. These are all activities associated with the *connections* cluster.

The reflection cluster

The competencies in this cluster include an element of reflectiveness on the part of the student about the processes needed or used to solve a problem. They relate to students' abilities to plan solution strategies and implement them in problem settings that contain more elements and may be more "original" (or unfamiliar) than those in the *connections* cluster. In addition to the competencies described for the *connections* cluster, for the *reflection* cluster the competencies include the following:

1. *Thinking and reasoning:* This involves posing questions ("how do we find…?", "which mathematics is involved…?", "what are the essential aspects of the problem or situation…?") and understanding the corresponding kinds of answers (provided by tables, graphs, algebra, figures, specification of key points etc.); distinguishing between definitions, theorems, conjectures, hypotheses and assertions about special cases, and reflecting upon or actively articulating these distinctions; understanding and handling mathematical concepts in contexts that are new or complex; and understanding and handling the extent and limits of given mathematical concepts, and generalising results.

2. *Argumentation*. This involves simple mathematical reasoning, including distinguishing between proving and proofs and broader forms of argument and reasoning; following, assessing and constructing chains of mathematical arguments of different types; and using heuristics (*e.g.* "what can or cannot happen, or be the case, and why?", "what do we know, and what do we want to obtain?", "which properties are essential?", "how are the objects related?").

3. *Communication*. This involves understanding and expressing oneself orally and in writing about mathematical matters ranging from reproducing the names and basic properties of familiar objects, and explaining computations and their results (usually in more than one way), to explaining matters that include complex relationships, including logical relationships. It also involves understanding others' written or oral statements about such matters.

4. *Modelling*. This involves structuring the field or situation to be modelled; translating "reality" into mathematical structures in contexts that may be complex or largely different from what students are usually familiar with; interpreting back and forth between models (and their results) and "reality", including aspects of communication about model results: gathering information and data, monitoring the modelling process and validating the resulting model. It also includes reflecting through analysing, offering a critique, and engaging in more complex communication about models and modelling.

5. *Problem posing and solving*. This involves posing and formulating problems well beyond the reproduction of practised standard pure and applied problems in closed form; solving such problems by invoking and using standard approaches and procedures, but also more original problem solving processes in which connections are being made between different mathematical areas and modes of representation and communication (schemata, tables, graphs, words, pictures). It also involves reflecting on strategies and solutions.

6. *Representation*. This involves decoding, encoding and interpreting familiar and less familiar representations of mathematical objects; choosing and switching between different forms of representation of mathematical objects and situations, and translating and distinguishing between different forms of representation. It further involves the creative combination of representations and the invention of non-standard ones.

7. *Using symbolic, formal and technical language and operations*. This involves decoding and interpreting symbolic and formal language practised in unknown contexts and situations, and handling statements and expressions containing symbols and formulae, including using variables, solving equations and undertaking calculations. It also involves the ability to deal with complex statements and expressions and with unfamiliar symbolic or formal language, and to understand and to translate between such language and natural language.

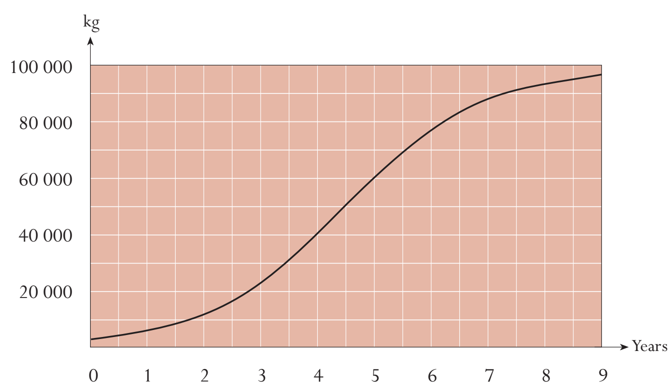
8. *Use of aids and tools*. This involves knowing about and using familiar or unfamiliar aids and tools in contexts, situations and ways that are quite different from those in which their use was introduced and practised. It also involves knowing about limitations of aids and tools.

Assessment items measuring the *reflection* cluster of competencies might be described with the following key descriptors: advanced reasoning, argumentation, abstraction, generalisation, and modelling applied to new contexts.

Examples of reflection cluster items

Mathematics Example 13: FISH GROWTH

Some fish were introduced to a waterway. The graph shows a model of the growth in the combined weight of fish in the waterway.



Suppose a fisherman plans to wait a number of years and then start catching fish from the waterway. How many years should the fisherman wait if he or she wishes to maximise the number of fish he or she can catch annually from that year on? Provide an argument to support your answer.

Mathematics Example 14: BUDGET

In a certain country, the national defence budget is $30 million for 1980. The total budget for that year is $500 million. The following year the defence budget is $35 million, while the total budget is $605 million. Inflation during the period covered by the two budgets amounted to 10 per cent.

A. You are invited to give a lecture for a pacifist society. You intend to explain that the defence budget decreased over this period. Explain how you would do this.

B. You are invited to lecture to a military academy. You intend to explain that the defence budget increased over this period. Explain how you would do this.

Source: de Lange and Verhage (1992). Used with permission.

It is clear that Example 13 fits the definition of mathematical problem solving in an authentic context. Students will have to come up with their own strategies and argumentation in a somewhat complex and unfamiliar problem. The complexity lies partly in the need to thoughtfully combine information presented both graphically and in text. Moreover, there is no answer that students can see immediately. They need to interpret the graph, and realise, for instance, that the growth rate reaches its maximum after five years or so. To be successful, students need to reflect on their solution as it emerges and think about the success of their strategy. Furthermore, the problem asks for an argument and an indication of "proof". One possibility is to use the trial and error method: see what happens if you wait only 3 years, for instance, and go on from there. If you wait until the end of the fifth year, you can have a big harvest every year – 20 000 kg of fish. If you can't wait that long, and start to harvest one year earlier, you can catch only 17 000 kg, and if you wait too long (six years), you can only catch 18 000 kg per year. The optimal result is therefore obtained when harvesting commences after five years.

Example 14 has been thoroughly researched with 16-year-old students (de Lange, 1987, pp. 87-90). It illustrates *reflection* cluster problems very well: the students recognised the literacy aspect immediately and quite often were able to do some kind of generalisation, as the heart of the solution lies in recognising that the key mathematical concepts here are absolute and relative growth. Inflation can of course be left out to make the problem more accessible to somewhat younger students without losing the key conceptual ideas behind the problem, but one loses in the complexity and thus in the required mathematisation. Another way to make the item "easier" is to present the data in a table or schema. These mathematisation aspects are then no longer necessary – students can start right away at the heart of the matter.

Figure 1.4 ■ **Diagrammatic representation of the competency clusters**

Mathematical literacy

The Reproduction Cluster	***The Connection Cluster***	***The Reflection Cluster***
• Standard representations and definitions • Routine computations • Routine procedures • Routine problem solving	• Modeling • Standard problem solving translation and interpretation • Multiple well-defined methods	• Complex problem-solving and posing • Reflection and insight • Original mathematical approach • Multiple complex methods • Generalisation

Summary of mathematical processes in OECD/PISA mathematics

Figure 1.4 provides a diagrammatic representation of the competency clusters and summarises the distinctions between them.

It would be possible to use the competency descriptions in the preceding pages to classify mathematics items and thereby to assign them to one of the competency clusters. One way to do this would be to analyse the demands of the item, then to rate each of the eight competencies for that item, according to which of the three clusters provided the most fitting description of item demands in relation to that competency. If any of the competencies were rated as fitting the description for the *reflection* cluster, then the item would be assigned to the *reflection* competency cluster. If not, but one or more of the competencies were rated as fitting the description for the *connections* cluster, then the item would be assigned to that cluster. Otherwise, the item would be assigned to the *reproduction* cluster, since all competencies would have been rated as fitting the competency descriptions for that cluster.

ASSESSING MATHEMATICAL LITERACY

Task characteristics

In the previous sections, the OECD/PISA mathematical literacy domain has been defined and the structure of the assessment framework has been described. This section considers in more detail features of the assessment tasks that will be used to assess students. The nature of the tasks and the item format types are described.

The nature of tasks for OECD/PISA mathematics

OECD/PISA is an international test of the literacy skills of 15-year-olds. All test items used should be suitable for the population of 15-year-old students in OECD countries.

In general, items include of some stimulus material or information, an introduction, the actual question and the required solution. In addition, for items with responses that cannot be automatically coded, a detailed coding scheme will is developed to enable trained markers across the range of participating countries to code the student responses in a consistent and reliable way.

In an earlier section of this framework, the situations to be used for OECD/PISA mathematics items were discussed in some detail. For OECD/PISA 2003, each item is set in one of four situation types: personal, educational/occupational, public and scientific. The items selected for the OECD/PISA 2003 mathematics instruments represent a spread across these situation types.

In addition, item contexts that can be regarded as *authentic* are preferred. That is, OECD/PISA values most highly tasks that could be encountered in real-world situations, and that have a context for which the use of mathematics to solve the problem would be authentic. Problems with extra-mathematical contexts that influence the solution and its interpretation are preferred as vehicles for assessing mathematical literacy.

Items should relate predominantly to the overarching ideas (the phenomenological problem categories) described in the framework. The selection of mathematics test items for OECD/PISA 2003 ensures that the four overarching ideas are well represented.

Items should embody one or more of the mathematical processes that are described in the framework, and should be identified predominantly with one of the competency clusters.

The level of reading required to successfully engage with an item is considered very carefully in the development and selection of items for inclusion in the OECD/PISA 2003 test instrument. The wording of items is as simple and direct as possible. Care is also taken to avoid question contexts that would create a cultural bias.

Items selected for inclusion in the OECD/PISA test instruments represent a broad range of difficulties, to match the expected wide ability range of students participating in the OECD/PISA assessment. In addition, the major classifications of the framework (particularly competency clusters and overarching ideas) should as far as possible be represented with items of a wide range of difficulties. Item difficulties are established in an extensive Field Trial of test items prior to item selection for the main OECD/PISA survey.

Item types

When assessment instruments are devised, the impact of the item type on student performance, and hence on the definition of the construct that is being assessed, must be carefully considered. This issue is particularly pertinent in a project such as OECD/PISA, in which the large-scale cross-national context for testing places serious constraints on the range of feasible item format types.

OECD/PISA will assess mathematical literacy through a combination of items with open constructed-response types, closed constructed-response types and multiple-choice types. About equal numbers of each of these item format types will be used in constructing the test instruments for OECD/PISA 2003.

Based on experience in developing and using test items for OECD/PISA 2000, the multiple-choice type is generally regarded as most suitable for assessing items that would be associated with the *reproduction* and *connections* competency cluster. For an example of this item type, see Example 15, which shows an item that would be associated with the *connections* competency cluster and with a limited number of defined response-options. To solve this problem, students must translate the problem into mathematical terms, devise a model to represent the periodic nature of the context described, and extend the pattern to match the result with one of the given options.

Mathematics Example 15: SEAL

A seal has to breathe even if it is asleep. Martin observed a seal for one hour. At the start of his observation the seal dived to the bottom of the sea and started to sleep. In 8 minutes it slowly floated to the surface and took a breath.

In 3 minutes it was back at the bottom of the sea again and the whole process started over in a very regular way.

After one hour the seal was:

A. at the bottom

B. on its way up

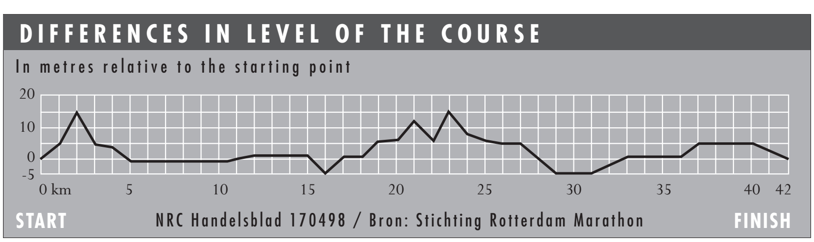
C. breathing

D. on its way down

For some of the higher-order goals and more complex processes, other item types will often be preferred. Closed constructed-response items pose questions similar to multiple-choice items, but students are asked to produce a response that can be easily judged to be either correct or incorrect. For items in this type, guessing is not likely to be a concern, and the provision of distractors (which influence the construct that is being assessed) is not necessary. For example, for the problem in Example 16 there is one correct answer and many possible incorrect answers.

Mathematics Example 16: Rotterdam Marathon

Tegla Loroupe won the 1998 marathon of Rotterdam. "It is easy", she said, "the course was quite flat". Here you see a graph of the differences in elevation of the Rotterdam marathon course:



What was the difference between the highest and the lowest points of the course?

_______________ m

Open constructed-response items require a more extended response from the student, and the process of producing a response frequently involves higher-order cognitive activities. Often such items not only ask the student to produce a response, but also require the student to show the steps taken or to explain how the answer was reached. The key feature of open constructed-response items is that they allow students to demonstrate their abilities by providing solutions at a range of levels of mathematical complexity, exemplified in Example 17.

Mathematics Example 17: INDONESIA

Indonesia lies between Malaysia and Australia. Some data of the population of Indonesia and its distribution over the islands is shown in the following table:

Region	Surface area (Km^2)	Percentage of total area	Population in 1980 (millions)	Percentage of total population
Java/Madura	132 187	6.95	91 281	61.87
Sumatra	473 606	24.86	27 981	18.99
Kalimantan (Borneo)	539 460	28.32	6 721	4.56
Sulawesi (Celebes)	189 216	9.93	10 377	7.04
Bali	5 561	0.30	2 470	1.68
Irian Jaya	421 981	22.16	1 145	5.02
TOTAL	1 905 569	100.00	147 384	100.00

One of the main challenges for Indonesia is the uneven distribution of the population over the islands. From the table we can see that Java, which has less than 7% of the total area, has almost 62% of the population.

Design a graph (or graphs) that shows the uneven distribution of the Indonesian population.

Source: de Lange and Verhage (1992). Used with permission.

For OECD/PISA, about one third of the mathematics items will be open constructed-response items. The responses to these items require coding by trained people who implement a coding rubric that may require an element of professional judgement. Because of the potential for disagreement between markers of these items, OECD/PISA will implement marker reliability studies to monitor the extent of disagreement. Experience in these types of studies shows that clear coding rubrics can be developed and reliable scores can be obtained.

OECD/PISA will make some use of a unit format in which several items are linked to common stimulus material. Tasks of this format give students the opportunity to become involved with a context or problem by asking a series of questions of increasing complexity. The first few questions are typically multiple-choice or closed constructed-response items, while subsequent ones are typically open constructed-response items. This format can be used to assess each of the competency clusters.

One reason for the use of common stimulus task formats is that it allows realistic tasks to be devised and the complexity of real-life situations to be reflected in them. Another reason relates to the efficient use of testing time, cutting down on the time required for a student to "get into" the subject matter of the situation. The need to make each scored point independent of others within the task is recognised and taken into account in the design of the OECD/PISA tasks and of the response coding and scoring rubrics. The importance of minimising bias that may result from the use of fewer situations is also recognised.

Assessment structure

The OECD/PISA 2003 test instruments contain a total of 210 minutes of testing time. The selected test items are arranged in seven clusters of items, with each item-cluster representing 30 minutes of testing time. The item-clusters are placed in test booklets according to a rotated test design.

The total testing time for mathematics is distributed as evenly as possible across the four overarching ideas (*quantity*, *space and shape*, *change and relationships*, and *uncertainty*), and the four situations described in the framework (*personal*, *educational / occupational*, *public*, and *scientific*). The proportion of items reflecting the three competency clusters (*reproduction*, *connections* and *reflection*) is about 1:2:1. About one-third of the items is in multiple-choice format type, about one-third in closed constructed-response type, and about one-third in open constructed-response type.

Reporting mathematical proficiency

To summarise data from responses to the OECD/PISA test instruments, a five-level described performance scale will be created (Masters & Forster, 1996; Masters, Adams, & Wilson, 1999). The scale will be created statistically, using an item response modelling approach to scaling ordered outcome data. The overall scale will be used to describe the nature of performance by classifying the student performances of different countries in terms of the five described performance levels, and thus provide a frame of reference for international comparisons.

Consideration will be given to developing a number of separate reporting scales. Such sub-scales could most obviously be based on the three competency clusters, or on the four overarching ideas. Decisions about the development of separate reporting scales will be made on a variety of grounds, including psychometric considerations, following analysis of the data generated by the OECD/PISA assessments. To facilitate these possibilities, it will be necessary to ensure that sufficient items are selected for inclusion in the OECD/PISA test instrument from each potential reporting category. Moreover, items within each such category will need to have a suitably wide range of difficulties.

The competency clusters described earlier in this framework reflect conceptual categories of broadly increasing cognitive demand and complexity, but do not strictly reflect a hierarchy of student performances based on item difficulty. Conceptual complexity is only one component of item difficulty that influences levels of performance. Others include familiarity, recent opportunity to learn and practice, etc. Thus, a multiple-choice item involving competencies from the *reproduction* cluster (for example, "which of the following is a rectangular parallelepiped?" followed by pictures of a ball, a can, a box, and a square) may be very easy for students who have been taught the meaning of these terms, but very difficult for others because of lack of familiarity with the terminology used. While it is possible to imagine relatively difficult *reproduction* cluster items and relatively easy *reflection* cluster items, and as far as possible items with a range of difficulties within each cluster type should be included, one would expect a broadly positive relationship between competency clusters and item difficulty.

Factors that will underpin increasing levels of item difficulty and mathematical proficiency include the following:

- The kind and degree of interpretation and reflection needed. This includes the nature of demands arising from the problem context; the extent to which the mathematical demands of the problem are apparent or to which students must impose their own mathematical construction on the problem; and the extent to which insight, complex reasoning and generalisation are required.

- The kind of representation skills that are necessary, ranging from problems where only one mode of representation is used, to problems where students have to switch between different modes of representation or to find appropriate modes of representation themselves.

- The kind and level of mathematical skill required, ranging from single-step problems requiring students to reproduce basic mathematical facts and perform simple computation processes through to multi-step problems involving more advanced mathematical knowledge, complex decision-making, information processing, and problem solving and modelling skills.

- The kind and degree of mathematical argumentation that is required, ranging from problems where no arguing is necessary at all, through problems where students may apply well-known arguments, to problems where students have to create mathematical arguments or to understand other people's argumentation or judge the correctness of given arguments or proofs.

At the lowest described proficiency level, students typically carry out single-step processes that involve recognition of familiar contexts and mathematically well-formulated problems, reproducing well-known mathematical facts or processes, and applying simple computational skills.

At higher proficiency levels, students typically carry out more complex tasks involving more than a single processing step. They also combine different pieces

of information or interpret different representations of mathematical concepts or information, recognising which elements are relevant and important and how they relate to one another. They typically work with given mathematical models or formulations, which are frequently in algebraic form, to identify solutions, or they carry out a small sequence of processing or calculation steps to produce a solution.

At the highest proficiency level, students take a more creative and active role in their approach to mathematical problems. They typically interpret more complex information and negotiate a number of processing steps. They produce a formulation of a problem and often develop a suitable model that facilitates its solution. Students at this level typically identify and apply relevant tools and knowledge in an unfamiliar problem context. They likewise demonstrate insight in identifying a suitable solution strategy, and display other higher-order cognitive processes such as generalisation, reasoning and argumentation to explain or communicate results.

Aids and tools

The OECD/PISA policy with regard to the use of calculators and other tools is that students should be free to use them as they are normally used in school.

This represents the most authentic assessment of what students can achieve, and will provide the most informative comparison of the performance of education systems. A system's choice to allow students to access and use calculators is no different, in principle, from other instructional policy decisions that are made by systems and are not controlled by OECD/PISA.

Students who are used to having a calculator available to assist them in answering questions will be disadvantaged if this resource is taken away.

SUMMARY

The aim of the OECD/PISA study is to develop indicators that show how effectively countries have prepared their 15-year-olds to become active, reflective and intelligent citizens from the perspective of their uses of mathematics. To achieve this, OECD/PISA has developed assessments that focus on determining the extent to which students can use what they have learned.

This framework provides a definition of mathematical literacy, and sets the context for the assessment of mathematical literacy in 2003 that will permit OECD countries to monitor some important outcomes of their education systems. The definition of mathematical literacy chosen for this framework is consistent with those definitions for literacy in reading and scientific literacy, and with the OECD/PISA orientation of assessing students' capacities to become active and contributing members of society.

The major components of the mathematics framework, consistent with the other OECD/PISA frameworks, include contexts for the use of mathematics, mathematical content and mathematical processes, each of which flows directly out of the literacy definition. The discussions of context and content emphasise features of the problems that confront students as citizens, while the discussions of processes emphasise the competencies that students bring to bear to solve those problems. These competencies have been grouped into three so-called "competency clusters" to facilitate a rational treatment of the way complex cognitive processes are adressed within a structured assessment program.

The emphasis of the OECD/PISA mathematics assessments on using mathematical knowledge and understanding to solve problems that arise out of day-to-day experience embodies an ideal that is achieved to varying degrees in different education systems around the world. The OECD/PISA assessments attempt to provide a variety of mathematical problems with varying degrees of built-in guidance and structure, but pushing towards authentic problems where students must do the thinking themselves.

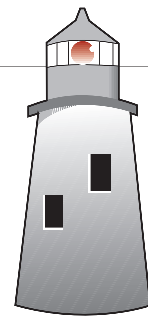
ADDITIONAL EXAMPLES

In this section, a number of mathematics items are presented in order to illustrate aspects of the OECD/PISA mathematics framework. The items are accompanied by commentary that is intended to describe elements of the items in relation to the framework.

This is the third set of sample mathematics items provided by the OECD. Seven units (comprising a total of 14 items) were published in *Measuring Student Knowledge and Skills* (OECD, 2000). A further five units (comprising a total of 11 items) were published in *Sample Tasks from the PISA 2000 Assessment* (OECD, 2002a).

Thirteen complete units are included here, comprising a total of 27 items. Each of these items was used in the field trial in 2002, as part of the item development process for the 2003 PISA main study. For a variety of reasons, largely related to the need for a complex balance of features in constructing the final test instruments, these items could not be included in the main study item selection. Some of them have measurement properties that make them less than ideal for use in an international test, but they are nevertheless useful for illustrative purposes and possibly for classroom use.

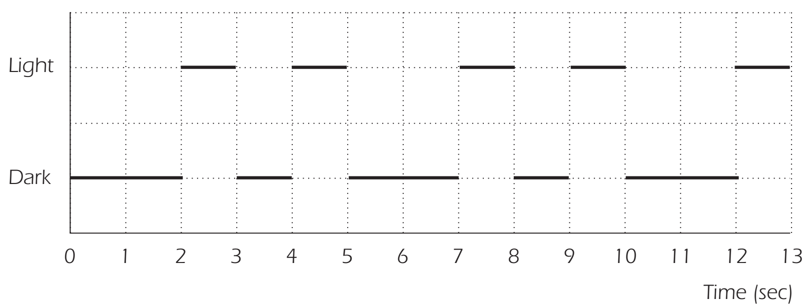
Mathematics Unit 1
LIGHTHOUSE



Lighthouses are towers with a light beacon on top. Lighthouses assist sea ships in finding their way at night when they are sailing close to the shore.

A lighthouse beacon sends out light flashes with a regular fixed pattern. Every lighthouse has its own pattern.

In the diagram below you see the pattern of a certain lighthouse. The light flashes alternate with dark periods.



It is a regular pattern. After some time the pattern repeats itself. The time taken by one complete cycle of a pattern, before it starts to repeat, is called the period. When you find the period of a pattern, it is easy to extend the diagram for the next seconds or minutes or even hours.

Mathematics Example 1.1

Which of the following could be the period of the pattern of this lighthouse?

A. 2 seconds.
B. 3 seconds.
C. 5 seconds.
D. 12 seconds

Scoring and comments on Mathematics Example 1.1

Full Credit

Code 1: Response C: 5 seconds.

No Credit

Code 0: Other responses.

Item type: *Multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Change and relationships*
Situation: *Public*

The unusual way this authentic problem is presented to the students immediately moves it beyond the *reproduction* competency cluster. The graphical representation will be new to most, if not all, students. This involves interpretation and reasoning skills right from the start of the problem. Most students will probably simulate the situation mentally: dark-dark-light-dark-light-dark-dark-light- and so on. They will have to find the "rhythm", either with the help of the graphical representation or with some other representation like the more word-oriented one just presented. This activity of making connections between different representations does make the problem fit into *connections* competency cluster.

The underlying concept of periodicity is important not only within the discipline of mathematics but also in daily life. The field trial suggests that most students do not find this problem very difficult, in spite of its unfamiliar appearance.

Some might argue that the context could favour students living near the sea or an ocean. It should be pointed out however that mathematical literacy includes the ability to use mathematics in contexts different from the local one. That does not necessarily mean that students living close to the sea might not be in a somewhat advantaged position. However, the item by country analysis gives no indication that this is the case here: landlocked countries did not perform differently from countries bordering on oceans.

Mathematics Example 1.2

For how many seconds does the lighthouse send out light flashes in 1 minute?

A. 4
B. 12
C. 20
D. 24

Scoring and comments on Mathematics Example 1.2

Full Credit

Code 1: Response D: 24.

No Credit

Code 0: Other responses.

Item type: *Multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Change and relationships*
Situation: *Public*

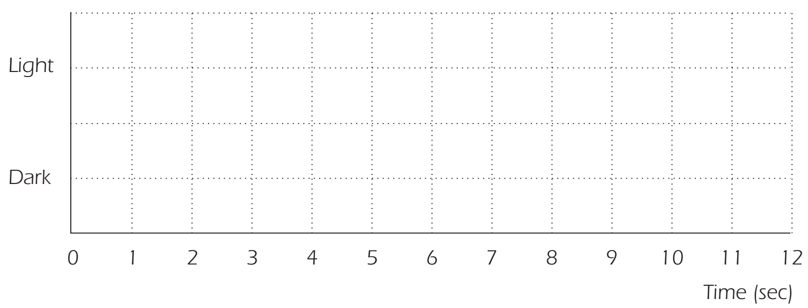
This example is slightly more difficult than Example 1.1, and the problem is also somewhat different in nature. The students have to translate and extend the visual model provided to a numeric model that helps them analyse the periodic pattern over a minute. It is not necessary that students have answered Example 1.1 correctly, but using that result is one possible strategy: since the period is 5, there are 12 periods in a minute, and since each period has 2 light flashes; the answer must be 24.

Another strategy that students at this level can use is to look at the graph for either the first 10 or the first 12 seconds, since these are numbers by which you can divide 60. If they look at 10 seconds, they will see 4 light flashes, to be multiplied by 6, and indeed the answer will again be 24. However, we do not really have "proof" that they fully understood the problem. The same holds for 12 seconds: 4 light flashes times 5 will give the students 20, which is wrong. The difference is that by choosing 10, the students have exactly 2 periods, and by choosing 12, they do not have a multiple of the period.

An authentic problem, not too difficult, associated with the *connections* cluster also because of the multiple steps needed.

Mathematics Example 1.3

In the diagram below, make a graph of a possible pattern of light flashes of a lighthouse that sends out light flashes for 30 seconds per minute. The period of this pattern must be equal to 6 seconds.



Scoring and comments on Mathematics Example 1.3

Full Credit

Code 2: Answers in which the graph shows a pattern of light and dark with flashes for 3 seconds in every 6 seconds, and with a period of 6 seconds. This can be done in the following ways:

- 1 one-second flash and 1 two-second flash (and this can be shown in several ways), or
- 1 three-second flash (which can be shown in four different ways).
- If 2 periods are shown, the pattern must be identical for each period.

Partial Credit

Code 1: Answers in which the graph shows a pattern of light and dark with flashes for 3 seconds in every 6 seconds, but the period is not 6 seconds. If 2 periods are shown, the pattern must be identical for each period.

- 3 one-second flashes, alternating with 3 one-second dark periods.

No Credit

Code 0: Other answers.

***Item type**: Open constructed-response*
***Competency cluster**: Reflection*
***Overarching idea**: Change and relationships*
***Situation**: Public*

The wording of the problem already indicates how "open" the problem is: "make a graph of a *possible* pattern of light flashes". Although the question seems to be related rather closely to the previous two questions, the correct response rate of the students was considerably lower, which made this item "rather difficult".

It is interesting that students are actually requested to "construct" or "design" something, which seems an important aspect of mathematical literacy: using mathematical competencies not only in a passive or derived way, but constructing an answer. Solving the problem is not trivial, because there are two conditions to be satisfied: equal amounts of time light and dark ("30 seconds per minute"), and a period of six seconds. This combination makes it essential that students really get at the conceptual level of understanding periodicity – already an indication that we are dealing with the *reflection* competency cluster.

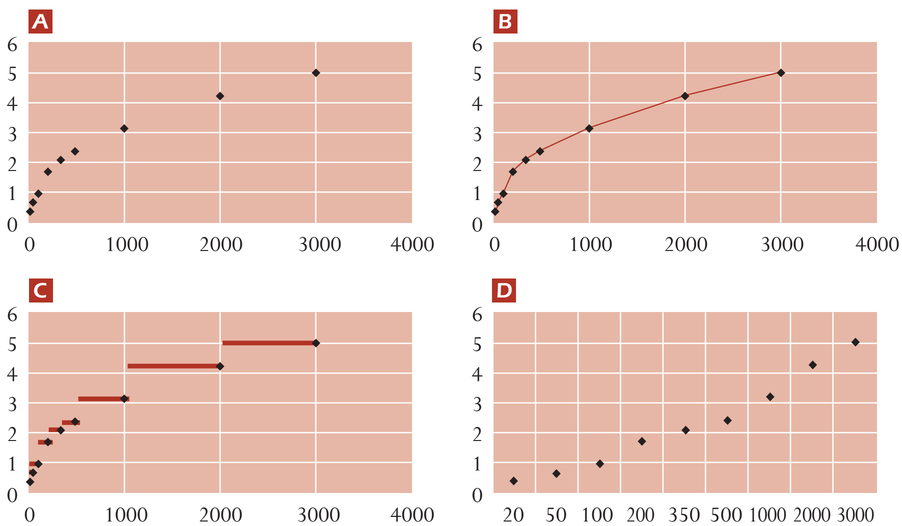
Mathematics Unit 2
POSTAL CHARGES

The postal charges in Zedland are based on the weight of the items (to the nearest gram), as shown in the table below:

Weight (to nearest gram)	Charge
Up to 20 g	0.46 zeds
21 g – 50 g	0.69 zeds
51 g – 100 g	1.02 zeds
101 g – 200 g	1.75 zeds
201 g – 350 g	2.13 zeds
351 g – 500 g	2.44 zeds
501 g – 1000 g	3.20 zeds
1001 g – 2000 g	4.27 zeds
2001 g – 3000 g	5.03 zeds

Mathematics Example 2.1

Which one of the following graphs is the best representation of the postal charges in Zedland?
(The horizontal axis shows the weight in grams, and the vertical axis shows the charge in zeds.)



Scoring and comments on Mathematics Example 2.1

Full Credit

Code 1: Response C

No Credit

Code 0: Other responses.

Item type: *Multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Uncertainty*
Situation: *Public*

The situation is clearly public and the problem is encountered frequently, but not necessarily in this form. In daily life citizens just hand in the postal package and ask what the postal charge will be. But informed citizens are expected to reflect a bit on the structure of the postal charges system and similar structures. Many people would know that the increase in postal charges is initially rather steep, but that with more weight, the increases get smaller. This structure is a rather common one.

To realise that such a structure can be made visual is quite another aspect. The graph is a "step graph", which students rarely, if at all, encounter in their school curriculum. This is probably the main reason why students found this problem quite difficult. Students are trained to connect the points in graphs, and sometimes they are wondering whether to connect the points with straight lines or with a nice curve – just like alternative B in the first example. B indeed seems to be a good answer, as it gives a price for every weight, unlike alternative A. The problem of course is that not all prices "exist", and that the range of prices is very limited: 0.46-0.69-1.02 and so on. Graph B is therefore incorrect. Graph C best fits the table of weights and charges as presented.

Another complicating factor in linking the table to a graph is the fact that graphs belonging to alternatives A, B and C all are very difficult to read for the first 500 grams because of the scales used. If students are really interested in the lowest values, alternative D might be appealing to them because it gives a very readable interpretation of the table – and the students might not even realise that the scale is not linear (horizontally). But if they realise the isolated points of the graph can never represent a structure as in the table, they will not even consider this option.

From the previous comments it may be clear that the competency cluster has to be *connections*, because of the unusual representation and the interpretation skills required to solve it.

Mathematics Example 2.2

Jan wants to send two items, weighing 40 grams and 80 grams respectively, to a friend.

According to the postal charges in Zedland, decide whether it is cheaper to send the two items as one parcel, or send the items as two separate parcels. Show your calculations of the cost in each case.

Scoring and comments on Mathematics Example 2.2

Full Credit

Code 1: Answers which specify that it will be cheaper to send the items as two separate parcels. The cost will be 1.71 zeds for two separate parcels, and 1.75 zeds for one single parcel containing both items.

No Credit

Code 0: Other answers.

***Item type**: Open constructed-response*
***Competency cluster**: Connections*
***Overarching idea**: Quantity*
***Situation**: Public*

This example is more practical than the previous one and was found to be relatively easy by the students during the field trial.

The example should be classified as belonging to the *connections* cluster, as the problem is not familiar to the students and requires just a bit more than reproduction competencies. Jan wants to send two items, 40 g and 80 g, to a friend. Although a bit counter-intuitive, the answer is easily found in the tables: 40 g cost 0.69 zeds, 80 g cost 1.02 zeds, so two parcels cost 1.71 zeds. One parcel weighing 120 g would cost 1.75 zeds to send. This is not mathematically complex, but is a relevant example of mathematical literacy, the kind of question that occurs in different situations in a citizen's life.

Mathematics Unit 3

HEARTBEAT

For health reasons people should limit their efforts, for instance during sports, in order not to exceed a certain heartbeat frequency.

For years the relationship between a person s recommended maximum heart rate and the person's age was described by the following formula:

Recommended maximum heart rate = 220 – age

Recent research showed that this formula should be modified slightly. The new formula is as follows:

Recommended maximum heart rate = 208 – (0.7 x age)

Mathematics Example 3.1

A newspaper article stated: "A result of using the new formula instead of the old one is that the recommended maximum number of heartbeats per minute for young people decreases slightly and for old people it increases slightly."

From which age onwards does the recommended maximum heart rate increase as a result of the introduction of the new formula? Show your work.

Scoring and comments on Mathematics Example 3.1

Full Credit

Code 1: Answers which specify 41 or 40.

$220 - age = 208 - 0.7 \times age$ results in $age = 40$, so people above 40 will have a higher recommended maximum heart rate under the new formula.

No Credit

Code 0: Other answers.

Item type: *Open constructed-response*
Competency cluster: *Connections*
Overarching idea: *Change and relationships*
Situation: *Public/Personal*

The classification of the situation depends of course on whether or not people are actually interested in data about their own health and body. One can safely argue that this item is somewhat scientific (because of the use of formulas) but many sportsmen and women (joggers, bicyclists, rowers, walkers, etc.) really do measure their heartbeat quite regularly during their exercises. More and more inexpensive instruments using micro-technology have made this aspect of human wellbeing much more accessible to ordinary people. This explains the situation classification as "Public/Personal."

Because we are really dealing more with modelling and less with trivial problem solving, a classification to the *connections* cluster seems rather straightforward, as well as the overarching idea *change and relationships*.

Comparing two formulas, even though they are merely rules of thumb, that relate to a person's well-being can be an intriguing activity, especially as they are partly presented as "word" formulas. This usually makes them more accessible to students. Even without asking a question, an initial reaction of students might be to see how their own age will lead to different recommended outcomes. As the PISA students are 15 years of age, the result under the old formula is 205 heartbeats per minute (realising that the information that the rate is *per minute* is not given), and under the new formula it is 198 (or 197). Thus they might already have found an indication that the statement in the newspaper article seems to be correct.

The example posed is somewhat more complex than this. It requires the students to find out when (at which age) the two formulas give the same result. This can be done by trial and error (a well established strategy by many students) but the more algebraic way seems more likely: $220 - a = 208 - (0.7 \times a)$, leading to an answer around 40.

From the viewpoint of mathematical literacy as well as from the viewpoint of more curricular oriented mathematics, this is quite an interesting and relevant problem. We note that the field trial data indicate that 15-year-old students found this problem quite difficult.

Mathematics Example 3.2

The formula *recommended maximum heart rate* = 208 – (0.7 x *age*) is also used to determine when physical training is most effective. Research has shown that physical training is most effective when the heartbeat is at 80% of the recommended maximum heart rate.

Write down a formula for calculating the heart rate for most effective physical training, expressed in terms of age.

Scoring and comments on Mathematics Example 3.2

Full Credit

Code 1: Answers which present any formula that is the equivalent of multiplying the formula for recommended maximum heart rate by 80 per cent.

- heart rate = 166 – 0.56 x age.
- heart rate = 166 – 0.6 x age.
- h = 166 – 0.56 x a.
- h = 166 – 0.6 x a.
- heart rate = (208 – 0.7 x age) x 0.8.

No Credit

Code 0: Other answers.

***Item type**: Open constructed-response*
***Competency cluster**: Connections*
***Overarching idea**: Change and relationships*
***Situation**: Public / Personal*

This example *seems* to measure exactly the same competencies as Example 3.1. The correct response rate is almost identical (during field trial). But there is a notable difference: in Example 3.1 students have to compare two formulas and to decide when they give the same result. In Example 3.2 the students are asked to "construct" a formula, something they are not frequently asked to do during their school career, in many countries. From a strictly mathematical viewpoint the question is not difficult at all: just multiply the formula by 0.8 – for instance, $\textit{Heart Rate} = (208 - 0.7 \times age) \times 0.8$. It would seem that even such simple manipulation of algebraic expressions, expressed in a practical and realistic context, presents a substantial challenge to many 15-year-olds.

Mathematics Unit 4

PAYMENTS BY AREA

People living in an apartment building decide to buy the building. They will put their money together in such a way that each will pay an amount that is proportional to the size of their apartment.

For example, a man living in an apartment that occupies one fifth of the floor area of all apartments will pay one fifth of the total price of the building.

Mathematics Example 4.1

Circle Correct or Incorrect for each of the following statements.

Statement	*Correct / Incorrect*
A person living in the largest apartment will pay more money for each square metre of his apartment than the person living in the smallest apartment.	Correct / Incorrect
If we know the areas of two apartments and the price of one of them we can calculate the price of the second.	Correct / Incorrect
If we know the price of the building and how much each owner will pay, then the total area of all apartments can be calculated.	Correct / Incorrect
If the total price of the building were reduced by 10%, each of the owners would pay 10% less.	Correct / Incorrect

Scoring and comments on Mathematics Example 4.1

Full Credit

Code 1: Answers which specify: Incorrect, Correct, Incorrect, Correct, in this order.

No Credit

Code 0: Any other combination of answers.

Item type: *Complex multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Change and relationships*
Situation: *Public*

The item demands quite a high level of competence in proportional reasoning, relating to a practical situation in society that is likely to be somewhat unfamiliar to 15-year-olds. The complex multiple-choice format used requires students to demonstrate a quite thorough understanding of the concepts involved. In addition, students are required to read and understand a series of complex mathematical propositions. The item was found to be quite difficult in the field trial.

Mathematics Example 4.2

There are three apartments in the building. The largest, apartment 1, has a total area of 95m^2. Apartments 2 and 3 have areas of 85m^2 and 70m^2 respectively. The selling price for the building is 300 000 zeds.

How much should the owner of apartment 2 pay? Show your work.

Scoring and comments on Mathematics Example 4.2

Full Credit

Code 2: Answers which specify 102 000 zeds, with or without the calculation shown. Unit not required.

- Apartment 2: 102 000 zeds
- Apt. 2: $\frac{85}{250}$ x 300 000 = 102 000 *zeds*
- $\frac{300\ 000}{250}$ = 1200 zeds for each square metre, so apartment 2 is 102 000.

Partial Credit

Code 1: Answers in which students applied the correct method, but present minor computational errors.

- Apt. 2: $\frac{85}{250}$ x 300 000 = 10 200 *zeds*

No Credit

Code 0: Other answers.

***Item type**: Open constructed-response*
***Competency cluster**: Connections*
***Overarching idea**: Quantity*
***Situation**: Public*

Example 4.2 is a more concrete example, involving "real" apartments, with "real" areas. The field trial results confirm that this question is considerably easier than the more abstract first question.

The competency cluster classification *connections* is appropriate given the multi-step problem solving, in an unfamiliar context, that is needed.

Mathematics Unit 5

STUDENT HEIGHTS

Mathematics Example 5.1

In a mathematics class one day, the heights of all students were measured. The average height of boys was 160 cm, and the average height of girls was 150 cm. Alena was the tallest – her height was 180 cm. Zdenek was the shortest – his height was 130 cm.

Two students were absent from class that day, but they were in class the next day. Their heights were measured, and the averages were recalculated. Amazingly, the average height of the girls and the average height of the boys did not change.

Which of the following conclusions can be drawn from this information?

Circle "Yes" or "No" for each conclusion.

Conclusion	*Can this conclusion be drawn?*
Both students are girls.	Yes / No
One of the students is a boy and the other is a girl.	Yes / No
Both students have the same height.	Yes / No
The average height of all students did not change.	Yes / No
Zdenek is still the shortest.	Yes / No

Scoring and comments on Mathematics Example 5.1

Full Credit

Code 1: Answers which specify "No" for all conclusions.

No Credit

Code 0: Any other combination of answers.

Item type: *Complex multiple-choice*
Competency cluster: *Reflection*
Overarching idea: *Uncertainty*
Situation: *Educational*

The classification is rather straightforward: *uncertainty*, since it requires understanding of statistical concepts; *educational*, since it is the kind of problem that one would only come across in a school setting; and *reflection*, because of the rather heavy "communications" aspect – students have to really understand the language, in detail, and the underlying concepts, which are rather sophisticated as well. The problem involves the ability to pose questions ("how do I know…?", "how do I find…?", "what can happen?", "what would happen if…?"), and the ability to understand and handle mathematical concepts (average) in contexts that are complex.

The mathematisation aspect of identifying the relevant mathematical content and information is important in this example. Superficial reading will lead to a misinterpretation. The situation is indeed complex: it varies within the class, and over time The entity "class" is used while discussing the average for *boys* and average for *girls* independently, but subsequently it is stated that Alena is the tallest (girl or student) and Zdenek the shortest (boy or student). The students have to read carefully to notice that Zdenek is a boy, which is essential, and Alena is a girl. The variation over time is that two students are not present initially, but when included in the measurement the next day the averages remain unchanged. The class thus gets bigger, but you don't know whether the two extra students are girls, boys or one of each.

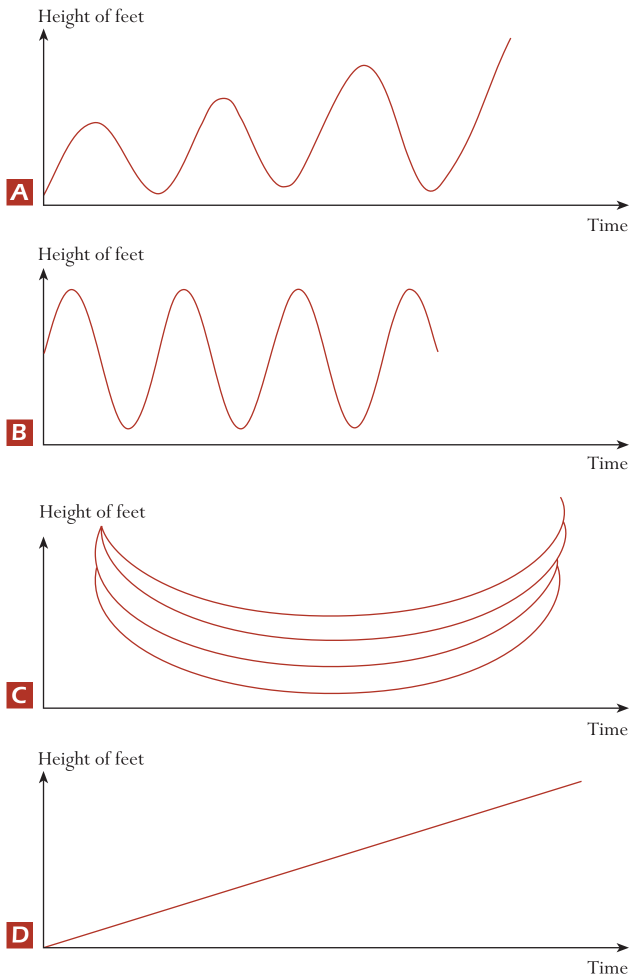
For students to get the five parts of this item correct, they need to explore in quite a sophisticated way the relationship between the data and the statistical summaries of those data. The field trial showed that this was an extremely challenging item for 15-year-olds.

Mathematics Unit 6
SWING

Mathematics Example 6.1

Mohammed is sitting on a swing. He starts to swing. He is trying to go as high as possible.

Which diagram best represents the height of his feet above the ground as he swings?



Scoring and comments on Mathematics Example 6.1

Full Credit

Code 1: Response A.

No Credit

Code 0: Other responses.

Item type: *Multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Change and relationships*
Situation: *Personal*

This kind of item is rather popular in some countries: which graphical representation fits the short story? In the 1970s the Canadian math educator Janvier promoted the format by asking students to identify a race track that would fit with the speed graph given – so more or less the other way around. A similar item was used in PISA 2000 and can be found in the publication *Sample Tasks from the PISA 2000 Assessment* (OECD, Paris, 2002a).

In the case of the swing, the question seems easier than the PISA 2000 item because one can easily dismiss certain alternatives almost immediately, which was certainly not the case with the race track problem.

Answer A seems to fit rather nicely. B does not start with feet low and does not get higher with each swing, C is just a visualisation of the swing action, and D does not swing. Answer A is therefore the most likely, which a majority of the students agree with.

The classification *connections* is appropriate since the students have to interpret and link at least two representations, textual and graphic, and link the best graph to the text. The familiarity of the context may bring a further practical component to the evaluation of the response options. Students have to understand the graph within the familiar context presented, but the graphic representations are not so familiar.

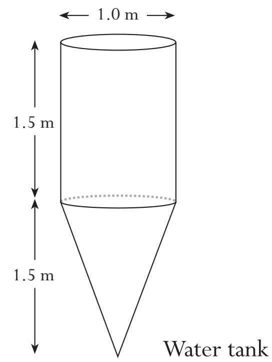
Mathematics Unit 7
WATER TANK

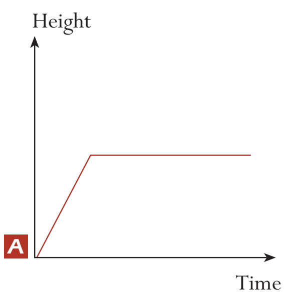
Mathematics Example 7.1

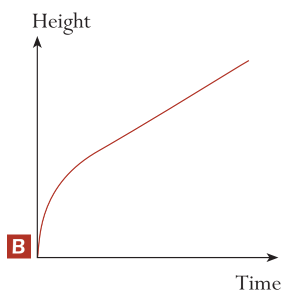
A water tank has shape and dimensions as shown in the diagram.

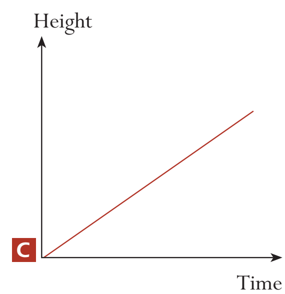
At the beginning the tank is empty. Then it is filled with water at the rate of one litre per second.

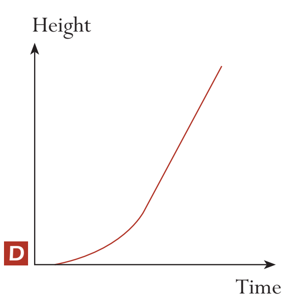
Which of the following graphs shows how the height of the water surface changes over time?

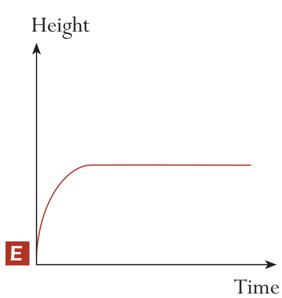












Scoring and comments on Mathematics Example 7.1

Full Credit

Code 1: Response B.

No Credit

Code 0: Other responses.

Item type: *Multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Change and relationships*
Situation: *Scientific*

This example is not very complex for students to understand: there is very little text and a clear diagram. Students are required to link the text and diagram, and then relate their understanding to graphical representations. These competencies fall into the *connections* cluster.

It is interesting to see that this item actually has quite a bit of redundant information. The measures of the tank are detailed, and the constant rate is described as one litre per second. But all this quantification does not help the students, since the graphs are "global" or "qualitative" only. This is interesting because one does not often see redundant information in mathematical items, but redundancy occurs almost every time one deals with problems in the real world. Actually, an important part of any mathematisation process is to identify the relevant mathematics and get rid of the redundant information.

Although the item is classified as having a scientific context, similar problems occur in personal situations. Filling a glass or vase or bucket, especially when the container is not cylindrical in shape, can cause some surprises if one does not take into account how the speed of the increase in height depends on the shape of the container, and such awareness fits under the definition of mathematical literacy.

Mathematics Unit 8

REACTION TIME

In a sprinting event, the "reaction time" is the time interval between the starter's gun firing and the athlete leaving the starting block. The "final time" includes both this reaction time, and the running time.

The following table gives the reaction time and the final time of 8 runners in a 100 metre sprint race.

Lane	Reaction time (sec)	Final time (sec)
1	0.147	10.09
2	0.136	9.99
3	0.197	9.87
4	0.180	Did not finish the race
5	0.210	10.17
6	0.216	10.04
7	0.174	10.08
8	0.193	10.13

Mathematics Example 8.1

Identify the Gold, Silver and Bronze medallists from this race. Fill in the table below with the medallists' lane number, reaction time and final time.

Medal	*Lane*	*Reaction time (sec)*	*Final time (sec)*
Gold			
Silver			
Bronze			

Scoring and comments on Mathematics Example 8.1

Full Credit

Code 1:

Medal	*Lane*	*Reaction time (sec)*	*Final time (sec)*
Gold	3	0.197	9.87
Silver	2	0.136	9.99
Bronze	6	0.216	10.04

No Credit

Code 0: Other answers.

***Item type**: Open constructed-response*
***Competency cluster**: Reproduction*
***Overarching idea**: Quantity*
***Situation**: Scientific*

A *reproduction* item that deals with understanding decimal notation (Quantity) but with some redundancy and complexity added because of the reaction time, which is not necessary to answer the first example. Almost two-thirds of the students that participated in the field trial came up with the correct answer, indicating that it is a relatively easy item for most 15-year-olds.

Mathematics Example 8.2

To date, no humans have been able to react to a starter's gun in less than 0.110 second.

If the recorded reaction time for a runner is less than 0.110 second, then a false start is considered to have occurred because the runner must have left before hearing the gun.

If the Bronze medallist had a faster reaction time, would he have had a chance to win the Silver medal? Give an explanation to support your answer.

Scoring and comments on Mathematics Example 8.2

Full Credit

Code 1: Answers which specify "yes", with an adequate explanation. For example:

- Yes. If he had a reaction time 0.05 sec faster, he would have equalled second place.
- Yes, he would have a chance to win the Silver medal if his reaction time were less than or equal to 0.166 sec.
- Yes, with the fastest possible reaction time he would have done a 9.93, which is good enough for the Silver medal.

No Credit

Code 0: Other answers, including answers which specify "yes" without an adequate explanation.

***Item type**: Open constructed-response*
***Competency cluster**: Connections*
***Overarching idea**: Quantity*
***Situation**: Scientific*

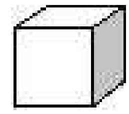
This example requires a moderate degree of verbal reasoning as well as mathematical reasoning. If one has answered Example 8.1 correctly, one sees clearly that Lane 6 (Bronze) is a slow starter (actually the slowest of all) and Lane 2 (Silver) a very fast starter (the fastest of them all), but they ended up with almost the same final time (different by only 0.05 seconds). The Lane 6 runner could thus have grabbed the Silver medal had his reaction been a bit faster, since the difference in their reaction times was quite a bit greater than the difference in final times.

Because of the interpretation skills needed, and the less trivial comparison of decimal numbers with different degrees of rounding, this item fits the *connections* competency cluster.

Mathematics Unit 9

BUILDING BLOCKS

Susan likes to build blocks from small cubes like the one shown in the following diagram:



Small cube

Susan has lots of small cubes like this one. She uses glue to join cubes together to make other blocks.

First, Susan glues eight of the cubes together to make the block shown in Diagram A:

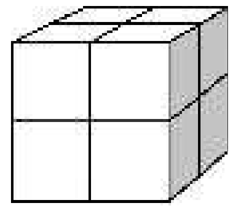


Diagram A

Then Susan makes the solid blocks shown in Diagram B and Diagram C below:

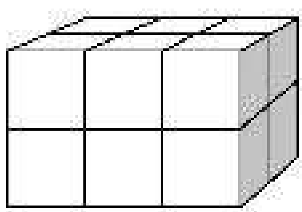


Diagram B

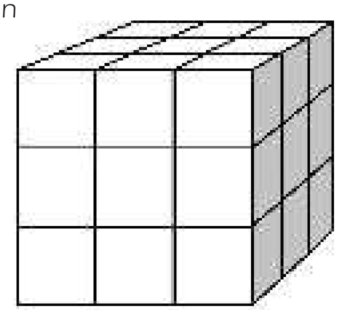


Diagram C

Mathematics Example 9.1

How many small cubes will Susan need to make the block shown in Diagram B?

Answer: ______________________ cubes.

Scoring and comments on Mathematics Example 9.1

Full Credit

Code 1: Answers which specify 12 cubes.

No Credit

Code 0: Other answers.

Item type: *Open constructed-response*
Competency cluster: *Reproduction*
Overarching idea: *Space and shape*
Situation: *Personal*

In every item bank it is compulsory to have really easy items as well as more difficult ones, as measured by students' results. This question is easy indeed: the students can imagine the problem easily, as they have probably been using these kinds of blocks often (Duplo, Lego, etc) and do not even need multiplication to get to the correct answer. For diagram B they see the first six cubes, and they know there are six cubes at the back. Both the familiarity as well as the simplicity makes this a clear *reproduction* item.

Mathematics Example 9.2

How many small cubes will Susan need to make the solid block shown in Diagram C?

Answer: ______________________ cubes.

Scoring and comments on Mathematics Example 9.2

Full Credit

Code 1: Answers which specify 27 cubes.

No Credit

Code 0: Other answers.

Item type: *Open constructed-response*
Competency cluster: *Reproduction*
Overarching idea: *Space and shape*
Situation: *Personal*

Example 9.2 differs from Example 9.1 in that the number of cubes is somewhat higher (27 instead of 12), but conceptually it is the same question. Field trial data show that students found this item relatively easy. This is to be expected because of the very basic competencies needed to solve this problem. Experts from participating countries also agreed that items such as this are close to their respective curricula.

Mathematics Example 9.3

Susan realises that she used more small cubes than she really needed to make a block like the one shown in Diagram C. She realises that she could have glued small cubes together to look like Diagram C, but the block could have been hollow on the inside.

What is the minimum number of cubes she needs to make a block that looks like the one shown in Diagram C, but is hollow?

Answer: ______________________ cubes.

Scoring and comments on Mathematics Example 9.3

Full Credit

Code 1: Answers which specify 26 cubes.

No Credit

Code 0: Other answers.

Item type: *Open constructed-response*
Competency cluster: *Connections*
Overarching idea: *Space and shape*
Situation: *Personal*

In Example 9.2 the assumption was that we were working with loose cubes and therefore needed 27, otherwise the block would collapse. If we are allowed to use glue, it may be possible to construct a block as depicted in C, but using fewer than 27 blocks. Although the "obvious" answer is 26 (take out the centre cube), more can be observed about this example. The problem is that the question does not explicitly state that Block C must look the same from all directions. This is relevant because one can take out more than one cube if one is allowed to use glue and has to stick to diagram C. However, it is *implicitly* stated by saying that the block has to be hollow on the *inside,* which takes care of this problem. From a language and interpretation point of view, however, this is not a straightforward question.

It can be classified as belonging to the *connections* cluster for several reasons: the mathematisation required to grasp the essentials of the question, the need to interpret Diagram C mentally as if it had a hole in it, the reasoning and thinking involved in order to reach the correct answer, and the lack of a standard procedure or algorithm.

Mathematics Example 9.4

Now Susan wants to make a block that looks like a solid block that is 6 small cubes long, 5 small cubes wide and 4 small cubes high. She wants to use the smallest number of cubes possible, by leaving the largest possible hollow space inside the block.

What is the minimum number of cubes Susan will need to make this block?

Answer: ______________________ cubes.

Scoring and comments on Mathematics Example 9.4

Full Credit

Code 1: Answers which specify 96 cubes.

No Credit

Code 0: Other answers.

Item type: *Open-Constructed Response*
Competency cluster: *Reflection*
Overarching idea: *Space and shape*
Situation: *Personal*

In Example 9.4 we need to assume (because of the way the problem is stated) that we can use glue again. The problem is now: "what is the minimum number of cubes necessary to build a hollow 4 x 5 x 6 block?"

As noted before, there are no standard problem- solving heuristics available to the students to answer this question. Having a mental image of one missing cube within a 3 x 3 x 3 building is quite another matter. Instead of having to mentally remove one cube, the students need to come up with a more generalisable strategy, involving more mathematical reasoning. It therefore makes sense to classify this item as belonging to the *reflection* competency cluster.

How can students find the right answer? A good strategy would be to start with the maximum number of cubes: 6 x 5 x 4 makes 120 altogether. Then mentally take out as many as possible from the centre. As it is 6 long, you can take out 4; as it is 5 wide, you can take out 3; as it is 4 high, you can take out 2. The total is 4 x 3 x 2, which equals 24. That gives 120-24 = 96, which is correct. It is a nice strategy, showing some real understanding. In a classroom situation, it might be interesting to ask the students for an explanation of their reasoning to discover useful teaching techniques.

Another strategy would be to look at the walls that are necessary to get the desired block. A picture might be helpful in this case.

To build the front wall we need 5 x 4 cubes; for the back wall, another 5 x 4 cubes. For the side wall we do not need 6 x 4 because we have already the front and back covered. The length of the side walls is therefore not 6 but 4, requiring 4 x 4 for each side. Finally we need to cover the bottom and top, leaving out what we have already. This gives us another 3 x 4. Total: 5 x 4; 5 x 4; 4 x 4; 4 x 4; 3 x 4; 3 x 4 – altogether, 96.

Undoubtedly, students will have different strategies at hand. A study like PISA could sometimes be used to find out which strategies are created or used by students when dealing with such a complex problem, where one has limited ways of representation in the traditional sense.

This is quite a challenging problem, almost strictly intra-mathematic, but still requiring competencies and skills, such as spatial visualisation, that are vital for literacy in mathematics.

Mathematics Unit 10

DRUG CONCENTRATIONS

Mathematics Example 10.1

A woman in hospital receives an injection of penicillin. Her body gradually breaks the penicillin down so that one hour after the injection only 60% of the penicillin will remain active.

This pattern continues: at the end of each hour only 60% of the penicillin that was present at the end of the previous hour remains active.

Suppose the woman is given a dose of 300 milligrams of penicillin at 8 o'clock in the morning.

Complete this table showing the amount of penicillin that will remain active in the woman's blood at intervals of one hour from 0800 until 1100 hours.

Time	0800	0900	1000	1100
Penicillin (mg)	300			

Scoring and comments on Mathematics Example 10.1

Full Credit

Code 2: Answers which include all three correct table entries, such as:

Time	0800	0900	1000	1100
Penicillin (mg)	300	180	108	64.8 or 65

Partial Credit

Code 1: Answers which include one or two correct table entries.

No Credit

Code 0: Other answers.

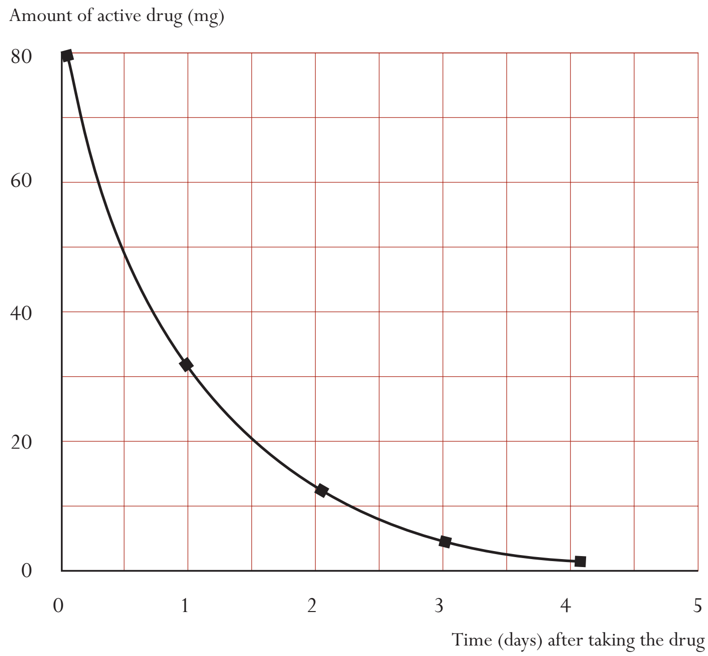
***Item type**: Open constructed-response*
***Competency cluster**: Connections*
***Overarching idea**: Change and relationships*
***Situation**: Scientific*

This first example may seem rather uncomplicated but exponential decay is not a trivial matter to many students. 60% of 60% of 60% of… may look like a simple rule but results on items like this one show that this is not the case. Although percentages are treated quite extensively in primary school, students are often not prepared to operationalise this knowledge in a different situation. To identify the relevant mathematical information means understanding the percentual or exponential decay (not necessarily understanding the expressions as such, but the concept), identifying the start-value (300) and applying the process repeatedly.

It is interesting that so many of the students tested in the field trial (50 per cent) failed to find the correct answer. This is important information for judging the quality and/or effectiveness of the teaching/learning process.

Mathematics Example 10.2

Peter has to take 80 mg of a drug to control his blood pressure. The following graph shows the initial amount of the drug, and the amount that remains active in Peter's blood after one, two, three and four days.



How much of the drug remains active at the end of the first day?

A. 6 mg.

B. 12 mg.

C. 26 mg.

D. 32 mg.

Scoring and comments on Mathematics Example 10.2

Full Credit

Code 1: Response D: 32 mg.

No Credit

Code 0: Other responses.

***Item type**: Multiple-choice*
***Competency cluster**: Reproduction*
***Overarching idea**: Change and relationships*
***Situation**: Scientific*

This example is easier than the previous one and actually requires nothing more than reading a graph, which leads to the conclusion that the item would require *reproduction* competencies. However, the item is set in a somewhat unusual context and some interpretation is needed.

Mathematics Example 10.3

From the graph for the previous question it can be seen that each day, about the same proportion of the previous day's drug remains active in Peter's blood.

At the end of each day which of the following is the approximate percentage of the previous day's drug that remains active?

A. 20%.

B. 30%.

C. 40%.

D. 80%.

Scoring and comments on Mathematics Example 10.3

Full Credit

Code 1: Response C: 40%.

No Credit

Code 0: Other responses.

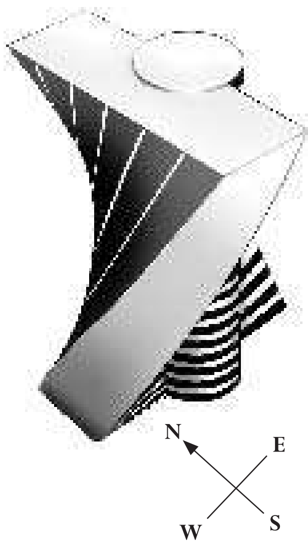
***Item type**: Multiple-choice*
***Competency cluster**: Connections*
***Overarching idea**: Change and relationships*
***Situation**: Scientific*

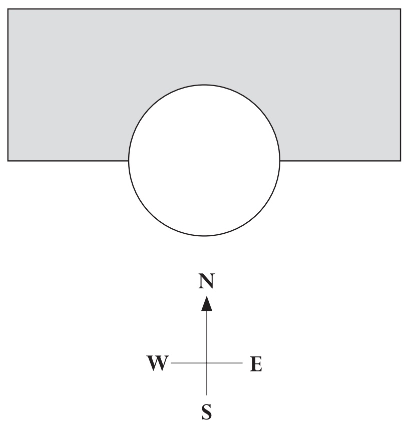
Example 10.3 relates to the graph presented with Example 10.2. The question is "what is the rate of decay?" in this particular situation. Presenting this question in multiple-choice format lets the students make an educated guess because they know the starting value, 80, and the next value – 32 (if they had answered Example 10.2 correctly) or around 30 (if they ignore Example 10.2 and go directly to the graph), and 3/8 is quite close to 40 per cent. The interpretation demands of the question places this item in the *connections* competency cluster.

Mathematics Unit 11
TWISTED BUILDING

In modern architecture, buildings often have unusual shapes. The picture below shows a computer model of a "twisted building" and a plan of the ground floor.

The compass points show the orientation of the building.





The ground floor of the building contains the main entrance and has room for shops. Above the ground floor there are 20 storeys containing apartments.

The plan of each storey is similar to the plan of the ground floor, but each has a slightly different orientation from the storey below. The cylinder contains the elevator shaft and a landing on each floor.

Mathematics Example 11.1

Estimate the total height of the building, in metres. Explain how you found your answer.

Scoring and comments on Mathematics Example 11.1

Full Credit

Code 2: Answers ranging from 50 to 90 metres, accompanied by a correct explanation. For example:

- One floor of the building has a height of about 2.5 meters. There is some extra room between floors. Therefore an estimate is 21 x 3 = 63 metres.
- Allow 4 m for each story. 20 of these equals 80 m, plus 10 m for the ground floor, which gives a total of 90 m.

Partial Credit

Code 1: Answers which present the correct calculation method and explanation, but using 20 stories instead of 21. For example:

- Each apartment could be 3.5 metres high; 20 stories of 3.5 metres gives a total height of 70 m.

No Credit

Code 0: Other answers, including answers without any explanation, answers with other incorrect numbers of floors, and answers with unreasonable estimates of the height of each floor (4 m would be the upper limit). For example:

- Each floor is around 5 m high, so 5 x 21 equals 105 metres.
- 60 m.

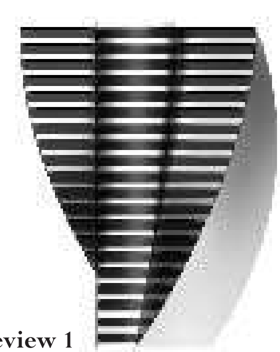
Item type*: Open constructed-response*
Competency cluster*: Connections*
Overarching idea: *Space and shape*
Situation*: Public*

The items in this unit require some imagination and insight, particularly in the area of spatial visualisation, in a public context that has familiar elements, but may seem novel to many students. The first example asks students to make some sensible judgments about what might be a reasonable height for each storey of a multi-storey building, including both the "visible" height of rooms in each storey and an allowance for the space needed between floors. Students need to carry out some elementary modelling, and to translate a visual representation into a numeric representation. These competencies are associated with the *connections* cluster.

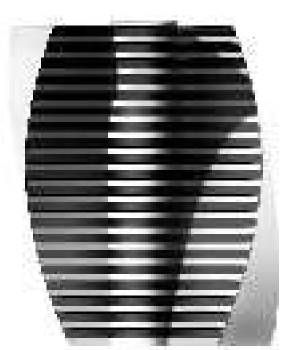
Many students in the field trial were able to do this, with the item slightly favouring boys. However, the item had a high omission rate, indicating that a good number of students were unwilling or unable to use their imagination in the required way.

Mathematics Example 11.2

The following pictures are sideviews of the twisted building.



Sideview 1



Sideview 2

From which direction has Sideview 1 been drawn?

A. From the North.

B. From the West.

C. From the East.

D. From the South.

Scoring and comments on Mathematics Example 11.2

Full Credit

Code 1: Response C: From the East.

No Credit

Code 0: Other responses.

***Item type**: Multiple-choice*
***Competency cluster**: Connections*
***Overarching idea**: Space and shape*
***Situation**: Public*

The second example asks students to mentally compare different visual representations of a building, and to choose from options that could describe the relationship between those representations. The spatial reasoning involved places the item in the *connections* cluster.

This item was considerably easier than the first one, but showed poor measurement properties in a number of participating countries. It may be that the quality of the graphic used in the field trial version was inadequate for the high visual demands of the item.

Mathematics Example 11.3

From which direction has Sideview 2 been drawn?

A. From the North West.

B. From the North East.

C. From the South West.

D. From the South East.

Scoring and comments on Mathematics Example 11.3

Full Credit

Code 1: Response D: From the South East.

No Credit

Code 0: Other responses.

Item type: *Multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Space and shape*
Situation: *Public*

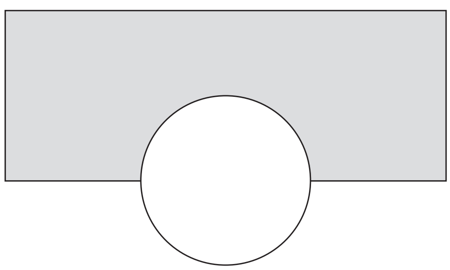
The third example is very similar to Example 11.2. It is interesting to note the different visual cues provided by the two "sideviews" used as stimuli for Examples 11.2 and 11.3 respectively. Example 11.3 was a little more difficult than Example 11.2, possibly because of the subtlety of the shadows in the stimulus, and the interpretation demands they impose.

Mathematics Example 11.4

Each storey containing apartments has a certain "twist" compared to the ground floor. The top floor (the 20th floor above the ground floor) is at right angles to the ground floor.

The drawing below represents the ground floor.

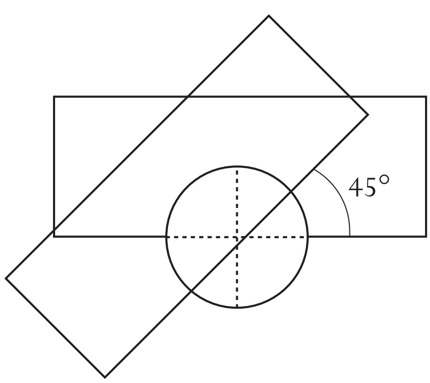
Draw in this diagram the plan of the 10th floor above the ground floor, showing how this floor is situated compared to the ground floor.



Scoring and comments on Mathematics Example 11.4

Full Credit

Code 2: Answers which present a correct drawing, meaning correct rotation point and anti-clockwise rotation. Accept angles from 40° to 50°.



Partial Credit

Code 1: Answers which incorrectly present either the rotation angle, the rotation point, or the rotation direction incorrect.

No Credit

Code 0: Other answers.

***Item type**: Open constructed-response*
***Competency cluster**: Connections*
***Overarching idea**: Space and shape*
***Situation**: Public*

The fourth example asks students to imagine the cumulative effect of the twisting phenomenon over a number of steps, and to construct a graphic representation of the 10^{th} floor. Again the spatial reasoning involved places this item in the *connections* cluster of competencies.

This item is relatively difficult, and again had quite a high omission rate in the field trial. It would seem that many 15-year-old students find this kind of geometric construction quite challenging.

Mathematics unit 12

ROCK CONCERT

Mathematics Example 12.1

For a rock concert a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.

Which one of the following is likely to be the best estimate of the total number of people attending the concert?

A. 2 000

B. 5 000

C. 20 000

D. 50 000

E. 100 000

Scoring and comments on Mathematics Example 12.1

Full Credit

Code 1: Response C: 20 000.

No Credit

Code 0: Other responses.

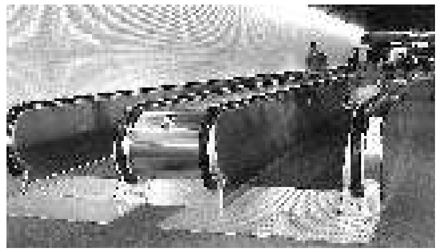
Item type: *Multiple-choice*
Competency cluster: *Connections*
Overarching idea: *Quantity*
Situation: *Public*

The mathematics framework highlights the importance of estimation skills as a part of the quantitative armoury of the mathematically literate citizen. This item is placed in a context that should be reasonably familiar to many 15-year-old students. However, after a small amount of interpretation, it requires students to take an active role in making assumptions about how much space (on average) people standing in a crowd might reasonably occupy. The kind of problem posing, and the mathematical reasoning it implies, places the item in the *connections* cluster.

Five response options were provided, so students had only to select the best option. Option A (2 000) implies that people would occupy on average 2.5 square meters, hardly a crowded concert. Option E (100 000) implies that on average there would be 20 people per square meter, barely possible and certainly not realistic. That leaves students to decide between three intermediate densities: 1 person, 4 people or 10 people per square meter. Which is more realistic under the conditions described (completely sold out, and the field was full with all the fans standing)? About 30 per cent of students in the field trial chose the most reasonable middle option C (20 000).

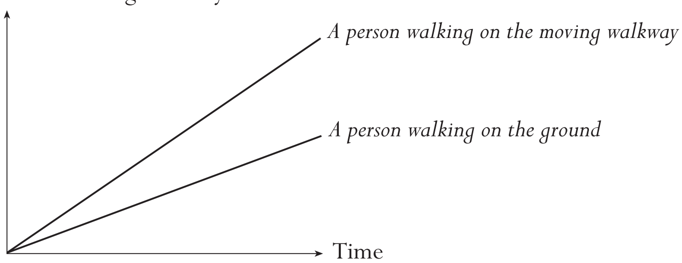
Mathematics Unit 13
MOVING WALKWAYS

Mathematics Example 13.1



On the right is a photograph of moving walkways.

The following Distance-Time graph shows a comparison between "walking on the moving walkway" and "walking on the ground next to the moving walkway."

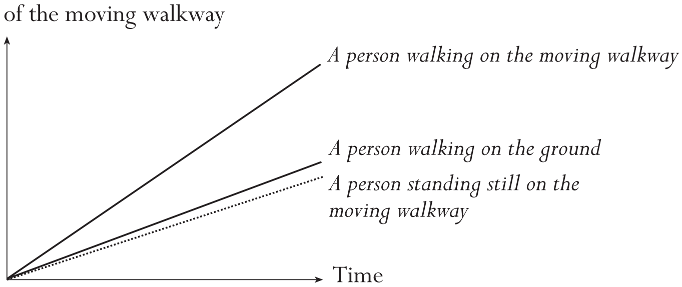


Assuming that, in the above graph, the walking pace is about the same for both persons, add a line to the graph that would represent the distance versus time for a person who is standing still on the moving walkway.

Scoring and comments on Mathematics Example 13.1

Full Credit

Code 1: Answers which show a line below the two lines, but it must be closer to the line of "a person walking on the ground" than to the baseline.



No Credit

Code 0: Other answers.

Item type: *Open constructed-response*
Competency cluster: *Reflection*
Overarching idea: *Change and relationships*
Situation: *Scientific*

The stimulus for this question depicts an object seen in some public places, and is also reminiscent of other similar phenomena that many 15-year-old students may be more familiar with (such as walking alongside a moving escalator, or running down stairs next to an elevator). However the nature of the question places this item in a "scientific" situation.

Students have to deal with a mathematical representation of the situation depicted, and must apply considerable imagination and insight to understand the representation. Quite sophisticated mathematical reasoning is then required to solve the problem and construct the appropriate response. These competencies are typical of the *reflection* cluster.

The item was found to be quite difficult by students in the field trial, with success rates around 15 per cent.

ELABORATION OF THE OVERARCHING IDEAS

Quantity

Description

In order to organise the world in which we live, there is a strong need for quantification: to express what is "big" or "small", "tall" or "short", "few" or "many", "more" or "less". We recognise patterns in the world around us while quantifying them: we call "fiveness" what collections of five apples, five people, five cars, five items have in common. The counting numbers 1, 2, 3, …are a way of capturing and describing those patterns. The counting numbers provide a starting point for calculating activities and a source for the search for deeper patterns like even and odd.

But the counting numbers might not be the earliest phenomenological encounter for young children. Children can recognise "small" and "big" in a qualitative way without attaching numbers to them, both with objects of different sizes (big biscuit vs. small biscuit) and with collections of objects (three objects vs. seven objects).

If a magnitude is measured, we see a further number use that is most important in everyday life. Length, area, volume, height, speed, mass, air pressure, money value are all quantified using measures.

Quantitative reasoning is an important aspect of dealing with quantities. It includes:

- number sense;
- understanding the meaning of operations;
- having a feel for the magnitude of numbers;
- elegant computations;
- mental arithmetic;
- estimations.

The "meaning of operations" includes the ability to perform operations involving comparisons, ratios and percentages. Number sense addresses issues of relative size, different representations of numbers, equivalent form of numbers, and using understanding of these things to describe attributes of the world.

Quantity also includes having a "feeling" for quantities and estimation. In order to be able to test numerical results for reasonableness, one needs a broad knowledge of quantities (measures) in the real world. Is the average speed of a car 5, 50 or 500 km/h? Is the population of the world 6 million, 600 million, 6 billion, or 60 billion? How tall is a tower? How wide is a river? The ability to make quick order-of-magnitude approximations is of particular importance, especially when viewed in light of the increasing use of electronic calculating tools. One needs to be able to see that 33 x 613 is something around 20 000. To achieve this skill one does not need extensive training in mental execution of traditional written algorithms, but a flexible and smart application of place value understanding and single-digit arithmetic (Fey, 1990).

Using number sense in an appropriate way, students can solve problems requiring direct, inverse, and joint proportional reasoning. They are able to estimate rates of change and provide a rationale for the selection of data and level of precision required by operations and models they use. They can examine alternative algorithms, showing why they work or in what cases they fail. They can develop models involving operations, and relationships between operations, for problems involving real-world data and numerical relations requiring operations and comparisons (Dossey, 1997).

In the overarching idea *quantity*, there is a place for "elegant" quantitative reasoning like that used by Gauss, as discussed in the following example. Creativity coupled with conceptual understanding should be valued at the level of schooling that includes 15-year-olds.

Examples

Gauss

Karl Friedrich Gauss's (1777-1855) teacher had asked the class to add together all the numbers from 1 to 100. Presumably the teacher's aim was to keep the students occupied for a time. But Gauss was an excellent quantitative reasoner and spotted a shortcut to the solution. His reasoning went like this:

You write down the sum twice, once in ascending order, then in descending order, like this:
1 + 2 + 3 + ... + 98 + 99 + 100
100 + 99 + 98 +............+ 3 + 2 + 1

Now you add the two sums, column by column, to give:
101 + 101 +..................+ 101 + 101

As there are exactly 100 copies of the number 101 in this sum its value is
100 x 101 = 10 100.

Since this product is twice the answer to the original sum, if you halve it, you obtain the answer: 5 050.

Triangular numbers

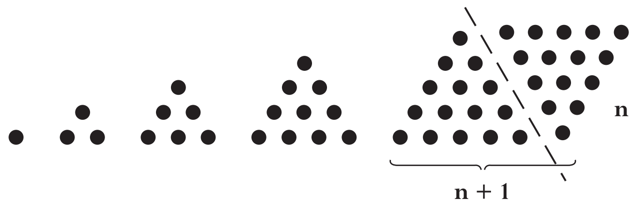
We might elaborate this example of quantitative thinking involving patterns of numbers a little further to demonstrate a link with a geometric representation of that pattern, by showing the formula that gives the general situation for Gauss's problem:

$$1 + 2 + 3 + \ldots + n = n(n + 1)/2$$

This formula also captures a geometric pattern that is well known: numbers of the form $n(n+1)/2$ are called triangular numbers, since they are exactly the numbers that are obtained by arranging balls in an equilateral triangle.

The first five triangular numbers 1, 3, 6, 10, 15 are shown below in Figure 1.5:

Figure 1.5 ■ **The first five triangular numbers**



Proportional reasoning

It will be interesting to see how students in different countries solve problems that lend themselves to the use of a variety of strategies. Differences can be expected especially in the area of proportional reasoning. In certain countries, mainly one strategy per item is likely to be used, while in other countries more strategies will be used. Also, similarities in reasoning will appear in solving problems that do not look very similar. This is in line with recent research results on TIMSS data (Mitchell, J. *et al.*, 2000). The following three items illustrate this point about different strategies and the relationships among them:

1. *Tonight you're giving a party. You want to buy 100 cans of soft drink. How many six-can packs are you going to buy?*
2. *A hang-glider with glide-ratio 1 to 22 starts from a sheer cliff at 120 metres. The pilot is aiming at a spot at a distance of 1 400 metres. Will she reach that spot (under conditions of no wind)?*
3. *A school wants to rent mini-vans (with seats for eight passengers) for going to a school camp and 98 students need transportation. How many vans does the school need?*

The first problem could be seen as a division problem (100 ÷ 6 = __) that then leaves the student with an interpretation problem back to the context (what is the meaning of the remainder?). The second problem can be solved by proportional reasoning (for every metre height I can fly a distance of 22 metres, so starting from 120 metres…). The third problem will be solved by many as a division problem. All three problems, however, can be solved using the ratio table method:

Bottles :	1	10	5	15	2	17
	6	60	30	90	12	102
Flying :	1	100	20	120		
	22	2 200	440	2 640		
Buses :	1	10	2	13		
	8	80	16	104		

Seeing this similarity is a skill that belongs to mathematical literacy: mathematically literate students do not need to look for the one available and appropriate tool or algorithm, but have available to them a wide array of strategies from which they can choose.

Percents

Carl went to a store to buy a jacket with a normal price of 50 zed that was on sale for 20% off. In Zedland there is a 5% sales tax. The clerk first added the 5% tax to the price of the jacket and then took 20% off. Carl protested: he wanted the clerk to deduct the 20% discount first and then calculate the 5% tax.

Does it make any difference?

Problems involving this kind of quantitative thinking, and the need to carry out the resulting mental calculations, are encountered frequently when shopping. The ability to effectively handle such problems is fundamental to mathematical literacy.

Space and shape

Description

Shape is a vital, growing and fascinating thing in mathematics, with strong ties to traditional geometry but going far beyond it in content, meaning and method. Interaction with real shapes involves understanding the visual world around us, its description, and encoding and decoding of visual information. It also means interpretation of visual information. In order to grasp the concept of shapes, students should be able to discover the way in which objects are similar and how they differ, to analyse the different components of the object, and to recognise shapes in different dimensions and representations.

It is important not to restrict ourselves to shapes as static entities. A shape can be transformed as an entity, and shapes can be modified. These changes can sometimes be visualised very elegantly using computer technology. Students should be able to see the patterns and regularities when shapes are changing. An example is shown in Figure 1.6 in the following section.

Another important dynamic aspect of the study of shapes is the relative position of shapes to each other in relation to the position of an observer. To achieve this we must not only understand the relative position of objects, but also consider questions of how and why we see things this way, etc. The relationship between shapes or images and their representation in both two and three dimensions plays a key role here.

Examples requiring this kind of thinking are abundant. Identifying and relating a photograph of a city to a map of that city and indicating from which point a picture was taken; the ability to draw a map; understanding why a building nearby looks bigger than a building further away; understanding how the rails of a railway track appear to meet at the horizon – all these questions are relevant for students within this overarching idea.

As students live in a three-dimensional space, they should be familiar with views of objects from three orthogonal aspects (for example the front, the side, and from above). They should be aware of the power and limitations of different representations of three-dimensional shapes as indicated by the example provided in the following Figure 1.7. Not only must they understand the relative position of objects, but also how they can navigate through space and through constructions and shapes. An example would be reading and interpreting a map and designing instructions on how to get from point A to point B using coordinates, common language or a picture.

Conceptual understanding of shapes also includes the ability to take a three-dimensional object and make a two-dimensional net of it, and vice-versa, even if the three-dimensional object is presented in two dimensions. An example of this is given in the following Figure 1.8.

In conclusion, here is a list of key aspects of *space and shape*:

- recognising shapes and patterns;
- describing, encoding and decoding visual information;
- understanding dynamic changes to shapes;
- similarities and differences;
- relative positions;
- 2-D and 3-D representations and the relations between them;
- navigation through space.

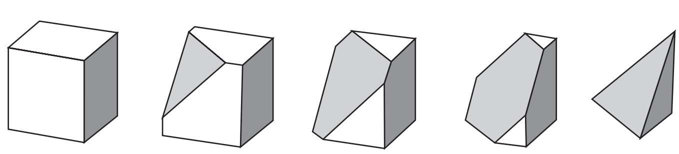
Examples

Figure 1.6 shows a simple example of the need for flexibility in seeing shapes as they change. It is based on a cube that is being "sectioned" (that is, plane cuts are made through the cube). A variety of questions could be asked, such as:

What shapes can be produced by one plane cut through a cube?

How many faces, edges, or vertices will be produced when a cube is sectioned in this way?

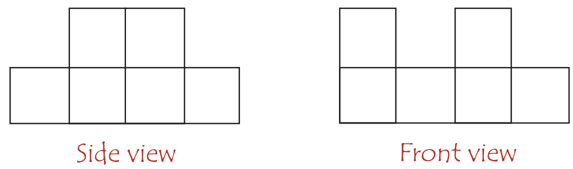
Figure 1.6 ■ **A cube, with plane cuts in various places**



Three examples of the need for familiarity with representations of three-dimensional shapes follow. In the first example, the side and front view of an object constructed of cubes is given in Figure 1.7. The question is:

How many cubes have been used to make this object?

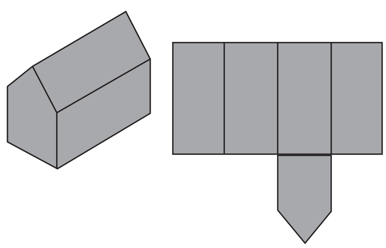
Figure 1.7 ■ **Side and front views of an object made from cubes**



It may come as a surprise to many – students and teachers alike – that the maximum number of cubes is 20 and the minimum is 6, (de Lange, 1995).

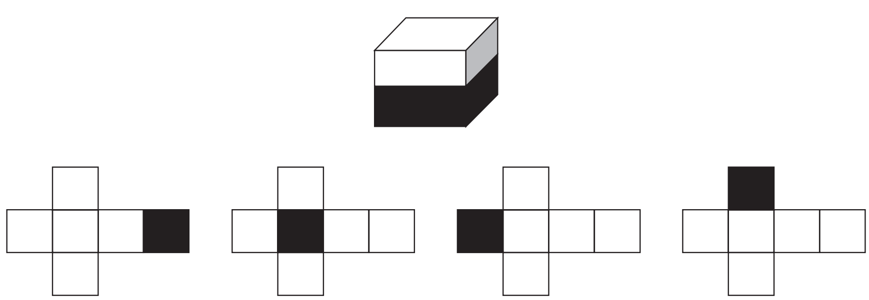
The next example shows a two-dimensional representation of a barn, and an incomplete net of the barn. The problem is to complete the net of the barn.

Figure 1.8 ■ **Two-dimensional representation of a three-dimensional barn, and its (incomplete) net**



A final example similar to the previous one is shown in the following Figure 1.9. (adapted from Hershkovitz *et al.*, 1996).

Figure 1.9 ■ **Cube with black bottom**



The lower half of the cube has been painted black. For each of the four nets, the bottom side is already black. Students could be asked to finish each net by shading the right squares.

Change and relationships

Description

In order to be sensitive to the patterns of change, Stewart (1990) states that we need to:

- represent changes in a comprehensible form;
- understand the fundamental types of change;
- recognise particular types of change when they occur;
- apply these techniques to the outside world;
- control a changing universe to our best advantage.

Change and relationships can be represented visually in a variety of ways: numerically (for example in a table), symbolically or graphically. Translation between these representations is of key importance, as is the recognition of an understanding of fundamental relationships and types of change. Students should be aware of the concepts of linear growth (additive process), exponential growth (multiplicative process) and periodic growth, as well as logistic growth, at least informally as a special case of exponential growth.

Students should also see the relationships among these models – the key differences between linear and exponential processes, the fact that percentage growth is identical with exponential growth, how logistic growth occurs and why, either in continuous or discrete situations.

Changes occur in a system of interrelated objects or phenomena where the elements influence each other. In the examples mentioned in the summary, all phenomena changed over time. But there are many examples in real life of matters in which objects are interrelated in a multitude of ways. For example:

If the length of the string of a guitar is halved, the new tone is an octave higher than the original tone. The tone is therefore dependent on the string length.

When we deposit money into a bank account, we know that the account balance will depend on the size, frequency and number of deposits and withdrawals, and the interest rates.

Relationships lead to dependency. Dependency concerns the fact that properties and changes of certain mathematical objects may depend on or influence properties and changes of other mathematical objects. Mathematical relationships often take the form of equations or inequalities, but relations of a more general nature may appear as well.

Change and relationships involves functional thinking. For 15-year-olds this includes students having a notion of rate of change, gradients and steepness (although not necessarily in a formal way), and dependence of one variable on another. They should be able to make judgements about how fast processes are taking place, also in a relative way.

This overarching idea closely relates to aspects of other overarching ideas. The study of patterns in numbers can lead to intriguing relationships: the study of Fibonacci numbers and the Golden Ratio are examples. The Golden Ratio is a concept that plays a role in geometry as well. Many more examples of *change and relationships* can be found in *space and shape*: the growth of an area in relation to the growth of a perimeter or diameter. Euclidean geometry lends itself also to the study of relationships. A well-known example is the relationship between the three sides of a triangle. If the length of two sides is known, the third is not determined, but the interval in which it lies is known: the interval's endpoints are the absolute value of the difference between the other two sides, and their sum, respectively. Several other similar relationships exist for the various elements of a triangle.

Uncertainty lends itself to various problems that can be viewed from the perspective of *change and relationships*. If two fair dice have been rolled and one of them shows four, what is the chance that the sum exceeds seven? The answer (50%) relies on the dependency of the probability at issue on the set of favourable outcomes. The required probability is the proportion of all such outcomes compared with all possible outcomes, which is a functional dependency.

Examples

School excursion

A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.

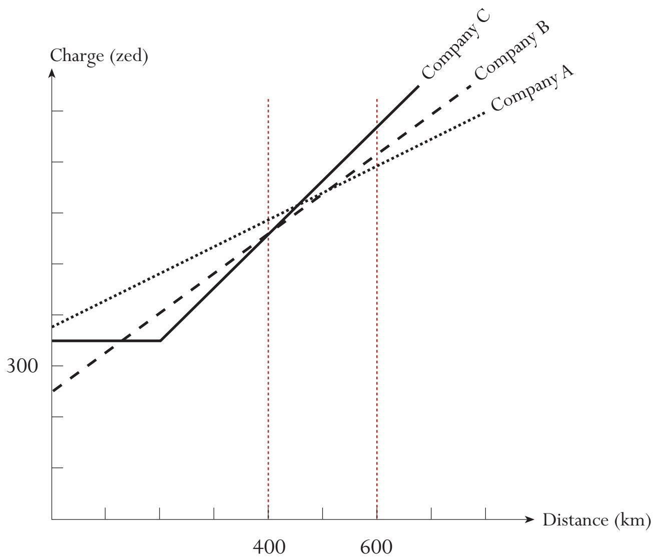
Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven.
Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven.
Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.

Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

Leaving aside the fictitious elements of the context, this problem could conceivably occur. Its solution requires the formulation and activation of several functional relationships, and equations and inequations. It can be dealt with by graphical as well as algebraic means, or combinations of both. The fact that the total travel distance in the excursion is not known exactly also introduces links to the *uncertainty* overarching idea.

A graphical representation of the problem is presented in the following Figure 1.10.

Figure 1.10 ■ **Excursion charges for three bus companies**



Cell growth

Doctors are monitoring the growth of cells. They are particularly interested in the day that the cell count will reach 60 000 because then they have to start an experiment. The table of results is:

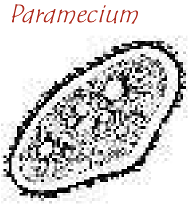
Time (days)	4	6	8	10	12	14	16	18	20
Cells	597	893	1 339	1 995	2 976	2 976	14 719	21 956	32 763

When will the number of cells reach 60 000?

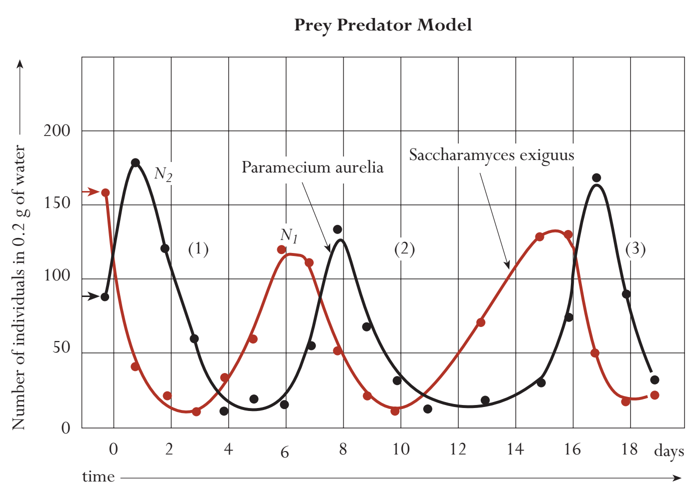
Prey-Predator

The following graph shows the growth of two living organisms – the Paramecium and Saccharomyces:

Paramecium



Saccharomyces



One of the two animals (predator) eats the other one (prey). Looking at the graph, can you judge which one is the prey and which one the predator?

One property of prey-predator phenomena is expressed as: The rate of growth of predators is proportional to the number of available prey. Does this property hold for the above graphs?

Uncertainty

Description

Science and technology rarely deal with certainty. The sciences are engaged in trying to find out how the world works, and to the degree that they succeed, so does our ability to describe with confidence what has happened in the past and to predict accurately what is likely to happen in the future. But scientific knowledge is rarely, if ever, absolute, not to mention sometimes being wrong, so there always remains some uncertainty in even the most scientific predictions.

The recommendations on the place of data, statistics and probability in school curricula emphasise data analysis. As a result, it is easy to view statistics in particular as a collection of specific skills. David S. Moore has pointed out what the overarching idea *uncertainty* really is all about. The OECD/PISA definition will follow his ideas as presented in *On the Shoulders of Giants* (Steen, 1990), and F. James Rutherford's as presented in *Why Numbers Count* (Steen, 1997).

The ability to deal intelligently with variation and uncertainty is the goal of instruction about data and chance. Variation is a concept that is hard to deal with: children who begin their education with spelling and multiplication expect the world to be deterministic; they learn quickly to expect one answer to be right and others wrong, at least when the answers take numerical form. Variation is unexpected and uncomfortable.

Statistics brings something to mathematics education that is unique and important: reasoning from uncertain empirical data. This kind of statistical thinking should be part of the mental equipment of every intelligent citizen. The core elements are:

- the omnipresence of variation in processes;
- the need for data about processes;
- the design of data production with variation in mind;
- the quantification of variation;
- the explanation of variation.

Data are not merely numbers, but numbers in a context. Data thus engage our knowledge of their context so that we can understand and interpret, rather than simply carry out arithmetical operations. Statistics in the early grades is taught not primarily for its own sake, but because it is an effective way to develop quantitative understanding and reasoning and to apply arithmetic and graphing to problem solving.

Collecting good data on important issues is no easy task. For the OECD/PISA study, the data must be interesting, relevant and practical, and carry a meaning for the students.

Data are obtained by measuring some characteristic, which means to represent it by a number. Thinking about measurement leads to a mature grasp of why some numbers are informative and others are irrelevant or nonsensical. First, what is a valid way to measure? Length is reasonably easy – a ruler will usually do it to a sufficient degree of accuracy for many purposes. But for area there may be a problem, since even for physical measurements uncertainty plays a role. Not only is the instrument important, but also the required degree of accuracy and the variability of measures.

The design of sample surveys is a core topic in statistics. Data analysis emphasises understanding the specific data at hand, assuming they represent a larger population. The concept of simple random samples is essential for 15-year-olds to understand the issues related to uncertainty.

A well known example:

In 1975, Ann Landers, a famous advice columnist, asked her readers:

"If you had to do it all over again, would you have children?"

10 000 people responded with 70% saying: NO.

It is well known that with voluntary responses, the overwhelming number come from people with strong (negative) feelings. A nationwide random sample about the same question rendered that 90% of the parents would like to have children again.

The essence of data analysis is to "let the data speak" by looking for patterns without first considering whether the data are representative of some larger universe.

Phenomena have uncertain individual outcomes and frequently the pattern of repeated outcomes is random. It has been shown that our intuition of chance profoundly contradicts the laws of probability (Garfield & Ahlgren, 1988; Tversky & Kahneman, 1974). This is in part due to the limited contact of students with randomness. The study of data offers a natural setting for such an experience. This explains why the priority of data analysis over formal probability and inference should be an important principle for the learning and teaching of uncertainty. Even at the college level, many students fail to understand probability and inference because of misconceptions that are not removed by study of formal rules. The concept of probability in the present OECD/PISA study will generally be based to situations regarding chance devices like coins, number cubes and spinners, or not too complex real-world situations that can be analysed intuitively, or can feasibly be modelled with these devices.

Uncertainty also appears from sources like natural variation in students' heights, reading scores, incomes of a group of people, etc. A step that is very important, even for 15-year-olds, is to see the study of data and chance as a coherent whole. One such principle is the progression of ideas from simple data analysis to data production to probability to inference.

The important specific mathematical concepts and activities in this area are:

- producing data – what are valid ways to measure particular characteristics and are the data valid for the proposed use? Critical attitude plays a very important role here, as does the design of the statistical study;
- data analysis and data display/visualisation, graphic representations of data, numerical descriptions like mean and median;
- probability;
- inference, which plays a minor role for students concerned in the study because formal treatment and specific methods are normally reserved for upper-grade secondary courses.

Examples

The following examples illustrate the *uncertainty* overarching idea.

Average age

If 40% of the population of a country are at least 60 years old, is it then possible for the average age to be 30?

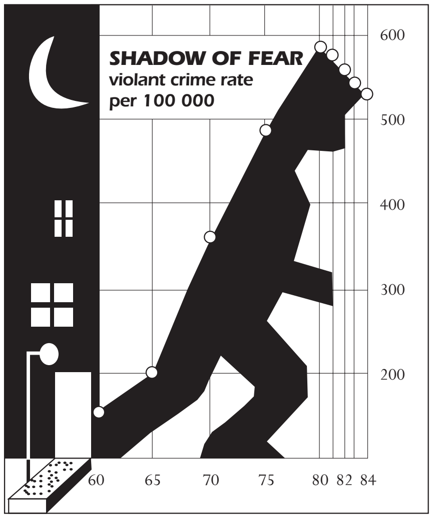
Growing incomes?

Has the income of people in Zedland gone up or down in recent decades? The median money income per household fell: in 1970 it was 34 200 zed, in 1980 it was 30 500 zed and in 1990 31 200 zed. But the income per person increased: in 1970 13 500 zed, in 1980 13 850, and in 1990 15 777 zed.

A household consists of all people living together at the same address. Explain how it is possible for the household income to go down at he same time the per-person income has risen in Zedland.

Rising crimes

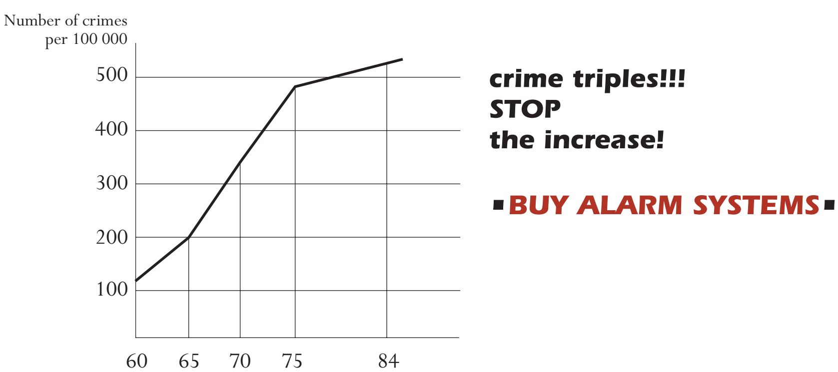
The following graph was taken from the weekly Zedland *News Magazine*:



It shows the number of reported crimes per 100 000 inhabitants, starting with five-year intervals, then changing to one-year intervals.

How many reported crimes per 100 000 were there in 1960?

Manufacturers of alarm systems used the same data to produce the following graph:



How did the designers come up with this graph and why?

The police were not too happy with the graph from the alarm systems manufacturers because the police want to show how successful crime fighting has been.

Design a graph to be used by the police to demonstrate that crime has decreased recently. ┘